MATH 569 CLASS NOTES

SIMON THOMAS' LECTURES ON SET THEORY

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CAUTION: these notes were typed up during the lectures, and so are probably full of typos along with misheard or misread parts (and perhaps genuine misunderstandings).

Section 1. *Polish Spaces, standard Borel spaces, and Borel equivalence relations*

1 • 1. Definition

A topological space $\langle X, \mathcal{T}\rangle$ is *Polish* iff it admits a complete separable metric d.

Notation: If $\langle X, \mathcal{T}\rangle$ is a topological space, then $\mathcal{B}(\mathcal{T})$ is the σ-algebra of Borel subsets of X. Normally when you talk about Borel equivalence relations, you don't care about the topology, but whether something is Borel.

1 • 2. Definition

Let $\mathcal{B}$ be a σ-algebra on a set X. Then $\langle X, \mathcal{B}\rangle$ is a *standard Borel space* iff there is a Polish topology $\mathcal{T}$ on X such that $\mathcal{B} = \mathcal{B}(\mathcal{T})$.

To what extent is the topology determined by the Borel subsets? The answer is not at all, by the following theorem, which has a proof to be found in Kechris' book.

1 • 3. Theorem

Let $\langle X, \mathcal{T}\rangle$ be a Polish space and let $Y \in \mathcal{B}(\mathcal{T})$. Then there exists a Polish topology $\mathcal{T}_Y \supseteq \mathcal{T}$ on X such that $\mathcal{B}(\mathcal{T}_Y) = \mathcal{B}(\mathcal{T})$ and Y is clopen in $\langle X, \mathcal{T}_Y\rangle$.

And this has a nice corollary.

1 • 4. Corollary

If $\langle X, \mathcal{B}\rangle$ is a standard Borel space, and $Y \in \mathcal{B}$, then $\langle Y, \mathcal{B}\restriction Y\rangle$ is a standard Borel space.

Here, $\mathcal{B}\restriction Y$ just means $\{Z \in \mathcal{B} : Z \subseteq Y\}$. This follows just because we may make Y into a clopen set, and $Y \subseteq X$ being closed implies that the corresponding metric is still complete.

Theorem 1 • 3 suggests that two standard Borel spaces should look alike. In fact, there is a unique uncountable one.

1 • 5. Theorem

There is a *unique* uncountable, standard Borel space up to isomorphism.

1 • 6. Definition

If X is a standard Borel space, then an equivalence relation E on X is *Borel* iff E is a Borel subset of $X \times X$.

So what we care about is comparing the relative complexity of Borel equivalence relations. Three examples we can define straight away are the following.

1. If X is a standard Borel space, then the identity relation $\mathrm{id}\restriction X = \mathrm{id}_X$ is Borel.

2. E_0 is the Borel equivalence relation of $2^{\mathbb{N}}$ defined by
$$x\ E_0\ y \leftrightarrow x(n) = y(n) \text{ for all by finitely many } n \in \mathbb{N}.$$
3. The Turing equivalence relation $\equiv_{\mathrm{T}}$ on $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ is defined by $A \equiv_{\mathrm{T}} B$ iff $A \leqslant_{\mathrm{T}} B$ and $B \leqslant_{\mathrm{T}} A$. And $\equiv_{\mathrm{T}}$ is a Borel equivalence relation.

1 • 7. Definition

A Borel equivalence relation E is *countable* iff every E-class is countable.

We want to compare the complexity of Borel equivalence relations by way of reduction. Now we don't allow arbitrary maps in the sense of reduction, but instead focus on Borel maps.

1 • 8. Definition

If X, Y are standard Borel spaces, a map $f : X \to Y$ is *Borel* iff graph(f) is a Borel subset of $X \times Y$.

There is an equivalent definition of Borel maps by the following theorem.

1 • 9. Theorem

If X, Y are standard Borel spaces, and $f : X \to Y$, then the following are equivalent:
1. f is Borel.
2. $f^{-1}"Z$ is Borel in X for each Borel subset $Z \subseteq Y$.

1 • 10. Definition

Suppose E, F are Borel equivalence relations on the standard Borel spaces X, Y.
1. E is *Borel reducible* to F, written $E \leqslant_{\mathrm{B}} F$, iff there is a Borel map $f : X \to Y$ such that
$$x\ E\ x' \leftrightarrow f(x)\ F\ f(x').$$
2. E and F are Borel bireducible, written $E \equiv_{\mathrm{B}} F$ iff $E \leqslant_{\mathrm{B}} F$ and $F \leqslant_{\mathrm{B}} E$.
3. $E <_{\mathrm{B}} F$ iff both $E \leqslant_{\mathrm{B}} F$ and $F \nleqslant_{\mathrm{B}} E$.

Let E, F be Borel equivalence relations on standard Borel spaces X, Y. $f : X \to Y$ is a *Borel homomorphism* from E to F iff $x\ E\ x' \to f(x)\ F\ f(x')$.

For example, if X and Y are uncountable, standard Borel spaces, then $\mathrm{id}_X \equiv_{\mathrm{B}} \mathrm{id}_Y$. As another example, we can define a Borel reduction $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $\mathrm{id}_{2^{\mathbb{N}}}$ to E_0 by $x \mapsto^{f} x\restriction 1^\frown x\restriction 2^\frown x\restriction 3^\frown \cdots {}^\frown x\restriction n^\frown \cdots$, because any difference between x and y occurs infinitely often in $f(x)$ and $f(y)$. As a challenge, find an explicit Borel reduction from E_0 to $\equiv_{\mathrm{T}}$.

The following is a theme of the course: Supposing that E and F are Borel equivalence relations, what techniques are available to prove that $E \nleqslant_{\mathrm{B}} F$? Often a proof that $E \nleqslant_{\mathrm{B}} F$ shows that if f is a Borel homomorphism from E to F, then there exists a "large set" which is mapped to a single F-class. So the idea is that if the reason is E is too complicated, then you'd expect the "kernel" to be large, which is similar to this statement. So what we need is suitable notions of largeness, and in this course, we will use three.

- **Category**

  1 • 11. Definition

  If X is a topological space, then $Z \subseteq X$ is *comeager* iff there exist dense, open subsets $\{D_n : n \in \omega\}$ such that $\bigcap_{n\in\omega} D_n \subseteq Z$.

  So if $\{Z_n : n \in \omega\}$ are comeager sets, then $\bigcap_{n\in\omega} Z_n$ is comeager. And this is supposed to motivate that this is a notion of largeness.

  Now if we want to separate $\mathrm{id}_{2^{\mathbb{N}}}$ and E_0, i.e. that $\mathrm{id}_{2^{\mathbb{N}}} <_{\mathrm{B}} E_0$. But it turns out that (provably) the notion of category can't be used any further. After category, we have measure.

- **Measure**

In particular, we look at probability measures.

1 • 12. Definition

Suppose $\langle X, \mathcal{B}\rangle$ is a standard Borel space. Then a *Borel probability measure* on X is a function $\mu : \mathcal{B} \to [0,1]$ such that
- $\mu(\emptyset) = 0$, $\mu(X) = 1$;
- If $A_n \in \mathcal{B}$ are disjoint, then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$

Remark: if $\{Z_n : n \in \omega\}$ satisfy $\mu(Z_n) = 1$, then $\mu(\bigcap_{n\in\omega} Z_n) = 1$. Note that this depends on the choice of measure.

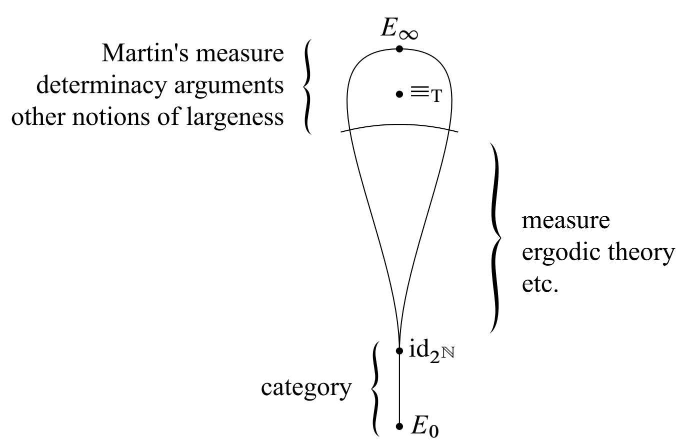
- **Martin's measure**

1 • 13. Definition

For each $A \in \mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$, the *corresponding cone* is
$$D_A = \{B \in \mathcal{P}(\mathbb{N}) : A \leqslant_{\mathrm{T}} B\}.$$

As you might expect by now, if $\{D_n : n \in \omega\}$ are cones, then there exists a cone $D \subseteq \bigcap_{n\in\omega} D_n$. To see this, choose $A_n \in \mathcal{P}(\mathbb{N})$ such that $D_n = D_{A_n}$. Then there exists an $A \in \mathcal{P}(\mathbb{N})$ such that $A_n \leqslant_{\mathrm{T}} A$ for all $n \in \omega$. Clearly $D_A \subseteq \bigcap_{n\in\omega} D_{A_n} = \bigcap_{n\in\omega} D_n$.

We get the following picture illustrating the limitations of these various notions of largeness. In particular, category is only useful in distinguishing E_0 from $\mathrm{id}_{2^{\mathbb{N}}}$. Measure and ergodic theory isn't able to deal with turing equivalence. It's also not yet known whether $E_\infty \equiv_{\mathrm{B}} \equiv_{\mathrm{T}}$.



1 • 14. Figure: Countable Borel Equivalence Relations

Section 2. Countable Borel Equivalence Relations: The Feldman–Moore Theorems

So far we've learned the language, and it's time we did more. As a matter of notation, write $G \curvearrowright X$ to indicate a group action.

2 • 1. Definition

If $G \curvearrowright X$, then E_G^X is the corresponding orbit equivalence relation:

$$x \; E_G^X \; y \leftrightarrow \exists g \in G(g \cdot x = y).$$

Suppose Γ is a countable group and X is a standard Borel space. An action $\Gamma \curvearrowright X$ is *Borel* iff for every $\gamma \in \Gamma$, the map $x \overset{\gamma}{\mapsto} \gamma \cdot x$ is Borel. In this case, we say X is a standard Borel Γ-*space*.

For example, if E has all of its classes as countable, then we can write $E = E_{\mathbb{Z}}^X$ for a suitable group action from $\mathbb{Z}$. Of course, to do this naïvely, we will use choice. But Feldman–Moore tells us that $E = E_\Gamma^X$ for some Γ and Borel group action. It turns out in the absence of choice that not every countable Borel equivalence relation can be realized by a group action from $\mathbb{Z}$.

The theorem itself is the source of a great many of connections between other branches of math.

2 • 2. Theorem (Feldman–Moore Theorem)

If E is a countable Borel equivalence relation on a standard Borel space X, then there exists a countable group Γ and a Borel action $\Gamma \curvearrowright X$ such that $E = E_\Gamma^X$.

The proof of Feldman–Moore Theorem (2 • 2) will use a suitable uniformization theorem. Suppose X and Y are any sets, and $P \subseteq X \times Y$. As a mater of notation,

- $x \in X$ has $P_x = \{y \in Y : \langle x, y\rangle \in P\}$;
- $y \in Y$ has $P^y = \{x \in X : \langle x, y\rangle \in P\}$; and
- $\mathrm{proj}_X(P) = \{x \in X : \exists y \in Y(\langle x, y\rangle \in P)\}$.

2 • 3. Definition

For $P \subseteq X \times Y$, $P^* \subseteq P$ is a uniformization iff for all $x \in \mathrm{proj}_X(P)$, there exists a unique $y \in Y$ with $\langle x, y\rangle \in P^*$.

So in essence, P^* is the graph of a function f where $f : \mathrm{proj}_X(P) \to Y$ such that $f(x) \in P_x$ holds. So AC implies every $P \subseteq X \times Y$ can be uniformized.

Now suppose that X, and Y are standard Borel spaces, and $P \subseteq X \times Y$ is Borel. Does P necessarily have a Borel uniformization? The answer is that it need not have one. For example, consider the following, due to Luzin.

2 • 4. Result

With $P \subseteq X \times Y$ and X, Y standard Borel spaces, if P has a Borel uniformization P^*, then $\mathrm{proj}_X(P)$ is Borel.

Proof Sketch .:.

Suppose that P^* is a Borel uniformization. Then $\mathrm{proj}_X : P^* \to X$ is injective, and so the image is Borel (by a famous theorem of Luzin). But $\mathrm{proj}_X(P) = \mathrm{proj}_X(P^*)$. ⊣

2 • 5. Corollary

There exists a Borel $P \subseteq X \times Y$ with X, Y standard Borel spaces, and with no Borel uniformization.

So in general, Borel sets in the plane don't have Borel uniformizations. But there are several cases where there are, as can be found (along with others) in Kechris' book. This is due to Luzin–Novikov.

2 • 6. Theorem

Suppose X, Y are standard Borel spaces and $P \subseteq X \times Y$ is a Borel subset such that P_x is countable (perhaps empty) for all $x \in X$. Therefore,

1. $\mathrm{proj}_X(P)$ is Borel, and P has a Borel uniformization; and
2. Moreover, we can express $P = \bigcup_{n\in\omega} P_n$, where each P_n is the Borel graph of a partial function; i.e. if $P_n(x, y)$ and $P_n(x, z)$, then $y = z$.

As an application of this, we get the following.

2 • 7. Corollary

Suppose that X, Y are standard Borel spaces and $f : X \to Y$ is a countable-to-one Borel map. Then $\mathrm{im}\, f$ is Borel, and there exists a Borel map $g : \mathrm{im}\, f \to X$ such that $f(g(y)) = y$ for all $y \in \mathrm{im}\, f$.

Proof .:.

Apply Theorem 2 • 6 to $P = \{\langle y, x\rangle : \langle x, y\rangle \in f\}$. $\dashv$

We will continue to use Corollary 2 • 7 with smoothness.

Section 3. *Smooth Countable Borel Equivalence Relations*

3 • 1. Theorem

If E is a Borel equivalence relation with uncountable many classes, then $\mathrm{id}_{2^{\mathbb{N}}} \leqslant_{\mathrm{B}} E$.

3 • 2. Definition

A Borel equivalence relation is *smooth* iff $E \leqslant_{\mathrm{B}} \mathrm{id}_Z$ for some (equivalently every) uncountable standard Borel space Z.

We immediately get the following observation from this definition.

3 • 3. Result

If E is a smooth, countable, Borel equivalence relation on an uncountable Borel space, then $E \equiv_{\mathrm{B}} \mathrm{id}_{2^{\mathbb{N}}}$.

We also get a nice, fairly easy theorem.

3 • 4. Theorem

If E is a countable Borel equivalence relation on a standard Borel space X, then the following are equivalent.

1. E is smooth.
2. There exists a Borel set $T \subseteq X$ which intersects every E-class in a unique point. (We say T is a *Borel transversal* for E.)
3. There exists a Borel map $s : X \to X$ such that $s(x)\ E\ x$, and if $x\ E\ y$ then $s(x) = s(y)$. (We say s is a *Borel selector* for E.)

Proof .:.

For (3) → (1), clearly s is a Borel reduction from E to id_X.

For (1) → (2), suppose $f : X \to Y$ is a Borel reduction from E to id_Y for some uncountable, standard Borel space Y. So f is a countable-to-one map. Hence $A = \mathrm{im}\, f$ is a Borel subset of Y, and there is an "inverse" $g : A \to X$ such that $f(g(a)) = a$ for all $a \in A$. Then g is injective and so $T = \mathrm{im}\, g$ is Borel and satisfies the desired property.

For (2) → (3), we can define a Borel selector by $s(x) = y$ iff $x\ E\ y$ and $y \in T$. $\dashv$

Note that this applies only to *countable* Borel equivalence relations. For example, there exists a smooth Borel equivalence relation with *no* Borel transversal. To see this, let X, Y, Z be such that X and Y are standard Borel spaces while $Z \subseteq X \times Y$ is Borel with $\mathrm{proj}_X(Z)$ non-Borel. Then Z is also a standard Borel space, and we can define a Borel equivalence relation E on Z by $\langle x, u\rangle \; E \; \langle y, v\rangle$ iff $x = y$. The map $\langle x, u\rangle \mapsto x$ is a Borel reduction from E to id_X, and so E is smooth. But any proposed $T \subseteq Z$ which is a Borel transversal for E has $\langle x, u\rangle \mapsto x$ as injective on T. Hence $\mathrm{proj}_X(T) = \mathrm{proj}_X(E) = \mathrm{proj}_X(Z)$ is Borel, contradicting construction.

Furthermore, there is a countable Borel equivalence relation which isn't smooth. At this point, this is easiest to show using measure, although an argument using category can be given.

3 • 5. Theorem

E_0 is not smooth

Proof .:.

Let μ be the usual uniform product probability measure on $2^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let $\pi_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel bijection $\langle x_0, \cdots, x_n, \cdots\rangle \overset{\pi_n}{\longmapsto} \langle x_0, \cdots, 1 - x_n, \cdots\rangle$, flipping just the nth entry.

Let $\Gamma = \bigoplus_{n\in\omega} C_n$, where $C_n = \langle \pi_n\rangle$. Then $\Gamma \curvearrowright \langle 2^{\mathbb{N}}, \mu\rangle$ as a group of measure-preserving transformations. Also clearly $E_0 = E_\Gamma^{2^{\mathbb{N}}}$. Furthermore, Γ acts freely on $2^{\mathbb{N}}$; i.e. if $1 \neq \gamma \in \Gamma$, then $\gamma \cdot x \neq x$ for all $x \in X$.

Suppose that E_0 is smooth. Then there exists a Borel transversal $T \subseteq 2^{\mathbb{N}}$. Since Γ acts freely, $2^{\mathbb{N}} = \bigsqcup_{\gamma\in\Gamma} \gamma"T$. This is because $\gamma_1 t_1 = \gamma_2 t_2$ implies $\gamma_2^{-1}\gamma_1 t_1 = t_2$. So as it's a transversal, $t_1 = t_2$. But as it act's freely, we get a contradiction.

Since T is Borel, T is μ-measurable. Since Γ preserves μ, $\mu(\gamma"T) = \mu(T)$ for all $\gamma \in \Gamma$. Hence
$$1 = \mu(2^{\mathbb{N}}) = \sum_{\gamma\in\Gamma} \mu(\gamma"T) = \sum_{\gamma\in\Gamma} \mu(T),$$
which is a contradiction ⊣

Let's return to Feldman–Moore Theorem (2 • 2). So more than just this, we have the following.

3 • 6. Theorem

Feldman–Moore Theorem (2 • 2) holds, and moreover Γ and $\Gamma \curvearrowright X$ can be chosen such that
$$x \; E \; y \leftrightarrow x = y \text{ or there exists } 1 \neq g \in \Gamma \text{ with } g^2 = 1 \text{ such that } g \cdot x = y$$

Proof .:.

Let E be a countable Borel equivalence relationon a standard Borel space X. Clearly we can suppose that X is uncountable. (Otherwise, the theorem is trivial.) Applying Theorem 2 • 6, since $E \subseteq X \times X$ has countable sections, we can express $E = \bigcup_{n\in\omega} F_n$ where each F_n is the Borel graph of a partial function. Out of this, we want to get a group action.

For each $n, m \in \mathbb{N}$, let $F_{n,m} = F_n \cap F_m^{-1}$ where $F^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in F\}$. Then each $F_{n,m}$ is the Borel graph of an injective partial function; and $E = \bigcup_{n,m} F_{n,m}$. Let $\Delta_x = \{\langle x, x\rangle : x \in X\}$.

Claim 1

We can express $X^2 \setminus \Delta_x = \bigcup_{p\in\mathbb{N}} A_p \times B_p$ where each A_p, B_p is a pair of disjoint Borel subsets.

Proof .:.

Since X is Borel isomorphic to $\mathbb{R}$, it suffices to work with $\mathbb{R}$. But this is easy for $\mathbb{R}$, since we just take sufficiently small discs with rational centers and radiuses. ⊣

For each n, m, p, let $F_{n,m,p} = F_{n,m} \cap (A_p \times B_p)$. So we're getting an injective function from a subset of A_p to a subset of B_p, and we can get the inverse just by going back. Explicitly, $F_{n,m,p}$ is the graph $f_{n,m,p}$ for some Borel bijection between disjoint Borel sets $D_{n,m,p}$ and $R_{n,m,p}$. Hence we can define a corresponding Borel bijection $g_{n,m,p}$ by

$$g_{n,m,p}(x) = \begin{cases} f_{n,m,p}(x) & \text{if } x \in D_{n,m,p} \\ f_{n,m,p}^{-1}(x) & \text{if } x \in R_{n,m,p} \\ x & \text{otherwise.} \end{cases}$$

Clearly $g_{n,m,p}^2 = 1$. Since $E \setminus \Delta_X = \bigcup_{n,m,p\in\omega} F_{n,m,p}$, we see that $\Gamma = \langle g_{n,m,p} : n, m, p \in \mathbb{N}\rangle$ satisfies our requirements. $\dashv$

Note that we can't witness this necessarily by finitely generated groups, nor necessarily by free groups. For example, $\mathbb{F}_2 \curvearrowright \mathcal{P}(\mathbb{F}_2)$ by $S \overset{g}{\mapsto} gS$ yields a universal relation, and the fact that it can't be realized by a free group uses something called Popa super-rigidity.

As a nice story, Thomas found this Popper's super-rigidity result on accident when searching something on Google, noticing that this was precisely what was needed for this idea about free groups. As he investigated who Popper was, he found that Popper was going to give a talk on the topic at UCLA, where Kechris works. And as it happened, Thomas was able to get to the result first.

Section 4. *Hyperfinite, and Universal Borel Equivalence Relations*

Why do we focus on countable Borel equivalence relations? One big reason is Theorem 3 • 6, suggesting connections with group theory, which yields lots of applications. In particular, the isomorphism relation on a class of (countable) structures. As it turns out, this is a countable Borel equivalence relation iff the structures are all "finitely generated" in a certain sense[i]. And then from here, it's a matter of thinking about how complicated these relations can get. To do this, it's useful to develop milestones.

4 • 1. Definition

A countable Borel equivalence relation E is *universal* iff $F \leqslant_{\mathrm{B}} E$ for every Borel equivalence relation F.

So by definition, for any two universal, countable Borel equivalence relation E, F, $E \equiv_{\mathrm{B}} F$. But this raises the question, does such a relation exist?

The following result by Friedman–Stanley shows that there is *no* universal Borel equivalence relation.

4 • 2. Theorem

If E is any Borel equivalence relation, then there exists a Borel equivalence relationE^+ such that $E <_{\mathrm{B}} E^+$.

So instead, consider the following definition.

4 • 3. Definition

Let $\mathbb{F}_2$ be the free group on 2 generators. Then E_∞ is the orbit equivalence relation of the Borel action $\mathbb{F}_2 \curvearrowright \mathcal{P}(\mathbb{F}_2)$ via $S \overset{g}{\mapsto} g"S = \{g \cdot s : s \in S\}$.

Equivalently, consider *the shift action* of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ defined by $(\gamma \cdot f)(x) = f(\gamma^{-1}x)$ for $f : \mathbb{F}_2 \to 2$.

Note that we have this γ^{-1} to preserve that $\delta \cdot (\gamma \cdot f)(x) = (\delta\gamma) \cdot f(x)$, as otherwise $\delta \cdot (\gamma \cdot f)(x) = (\gamma \cdot f)(\delta \cdot x) = f(\gamma \cdot \delta x)$ which might not be $f(\delta \cdot \gamma x)$.

These two actions are the same by the following: consider $f = \chi_S$. Then $(\gamma \cdot \chi_S)(x) = 1$ iff $\chi_S(\gamma^{-1}x) = 1$ iff $\gamma^{-1}x \in S$ iff $x \in \gamma S$. Hence $\gamma \cdot \chi_S = \chi_{\gamma"S}$.

[i] in the sense that fixing just finitely many elements gives only one transformation

We have the following target theorem.

4 • 4. Theorem (Target Theorem)

E_∞ is a countable Borel equivalence relation.

Now we look at a more general version of the shift action.

4 • 5. Definition

Let X be a standard Borel space and G a countable group. Then define $X^G = \{p : p : G \to X\}$ with the product Borel structure. The *shift action* $G \curvearrowright X^G$ is the Borel action $g \cdot p(h) = p(g^{-1} \circ h)$. The corresponding orbit equivalence relation is denoted $E(G, X)$.

For example, $E_\infty = E(\mathbb{F}_2, 2)$.

4 • 6. Proposition

Let G be a countable group and let X be a standard Borel G-space. Then $E_G^X \leqslant_B E(G, 2^{\mathbb{N}})$.

Proof .:.

Let $\{U_n : n \in \omega\}$ be a sequence of Borel subsets of X which separates points, meaning for any two points $x \neq y$, there are $n, m \in \omega$ with $x \in U_n \setminus U_m$ and $y \in U_m \setminus U_n$. Define the Borel map $f : X \to (2^{\mathbb{N}})^G$ by $[f(x)(g)](n) = 1$ iff $g^{-1}(x) \in U_n$.

We claim that f is a Borel reduction from E_G^X to $E(G, 2^{\mathbb{N}})$. Mostly this consists in going through the definitions. Firstly, suppose that $y = h \cdot x$ for some $h \in G$. Then

$$\begin{aligned} [f(y)(h)](n) = 1 \quad &\text{iff} \quad [f(hx)(g)](n) = 1 \\ &\text{iff} \quad g^{-1}hx \in U_n \\ &\text{iff} \quad (h^{-1}g)^{-1}x \in U_n \\ &\text{iff} \quad [f(x)(h^{-1}g)](n) = 1. \end{aligned}$$

Thus $f(y)(g) = f(x)(h^{-1}g)$ and so $f(y) = h \cdot f(x)$.

Conversely, suppose that $f(y) = h \cdot f(x)$ for some $h \in G$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} y \in U_n \quad &\text{iff} \quad [f(y)(1)](n) = 1 \\ &\text{iff} \quad [h \cdot f(x)(1)](n) = 1 \\ &\text{iff} \quad [f(x)(h^{-1})](n) = 1 \\ &\text{iff} \quad hx \in U_n. \end{aligned}$$

So as $\{U_n : n \in \omega\}$ separates points, $y = h \cdot x$. ⊣

4 • 7. Proposition

Let $\mathbb{F}_\omega$ be the free group on countably many generators. Then $E(\mathbb{F}_\omega, 2^{\mathbb{N}})$ is a universal countable Borel equivalence relation.

Proof .:.

Let E be a countable Borel equivalence relation on a standard Borel space X. By Theorem 3 • 6, there exists a countable group Γ and a Borel action $\Gamma \curvearrowright X$ such that E is the corresponding orbit equivalence relation E_Γ^X. Let $\pi : \mathbb{F}_\omega \to \Gamma$ be a surjective homomorphism. Then we can define a Borel action $\mathbb{F}_\omega \curvearrowright X$ by $x \overset{g}{\mapsto} \pi(g) \cdot x$. Clearly, $E = E_{\mathbb{F}_\omega}^X$. So by Proposition 4 • 6,

$$E = E_{\mathbb{F}_\omega}^X \leqslant_B E(\mathbb{F}_\omega, 2^{\mathbb{N}}) \dashv$$

4 • 8. Proposition

Suppose that G and H are countable groups and $\pi : H \to G$ is a surjective homomorphism. If X is any standard Borel space, then
$$E(G, X) \leqslant_B E(H, X)$$
via the Borel reduction $p \mapsto p^*$ where $p^*(h) = p(\pi(h))$.

The proof of this is left to the reader that wants to be complete. Intuitively, the result holds since H is bigger than G. Similarly, although we require a bit of a trick this time, we get the following proposition.

4 • 9. Proposition

Suppose G, H are countable groups and $G \leqslant H$. If X is a standard Borel space, then
$$E(G, X) \leqslant_B E(H, X).$$

Proof .:.

Fix some $x_0 \in X$ and consider the Borel map $f : X^G \to X^H$ by $p \mapsto p^*$ where
$$p^*(h) = \begin{cases} p(h) & \text{if } h \in G \\ x_0 & \text{if } h \notin G. \end{cases}$$
We claim that f is a Borel reduction from $E(G, X)$ to $E(H, X)$. Clearly if $g \cdot p = q$ for some $g \in G$, then $g \cdot p^* = q^*$. What we want is the converse.

Conversely, suppose that $h \cdot p^* = q^*$ for some $h \in H$. If $h \in G$, then clearly $h \cdot p = q$. So suppose $h \in H \setminus G$. In this case, we must have that for all $g \in G$, $q(g) = p(h^{-1}g) = x_0$ since $h^{-1}g \notin G$. This means $q^*(h) = x_0$ for all $h \in H$, and so $p^* = q^*$, and $p = q$. ⊣

Applying the (unproved) Target Theorem (4 • 4) and Proposition 4 • 9, we see that if G is a countable group with a free non-abelian subgroup, then $E_G^{\mathcal{P}(G)} \equiv_B E_\infty$.

4 • 10. Proposition

If G is a countable group, then $E(G, 2^{\mathbb{Z}\setminus\{0\}}) \leqslant_B E(G \times \mathbb{Z}, 3)$.

Proof .:.

Consider the Borel map $f : \left(2^{\mathbb{Z}\setminus\{0\}}\right)^G \to 3^{G\times\mathbb{Z}}$ by $p \mapsto p^*$ by
$$p^*(g, n) = \begin{cases} p(g)(n) & \text{if } n \neq 0 \\ 2 & \text{if } n = 2. \end{cases}$$
We claim that f is a Borel reduction from $E(G, 2^{\mathbb{Z}\setminus\{0\}})$ to $E(G \times \mathbb{Z}, 3)$. First, suppose that $p, q \in \left(2^{\mathbb{Z}\setminus\{0\}}\right)^G$ and that $g \cdot p = q$ for some $g \in G$. Then it is easily checked that $\langle g, 0\rangle p^* = q^*$.

Conversely, suppose that $q^* = \langle g, n\rangle p^*$ for some $\langle g, n\rangle \in G \times \mathbb{Z}$. If $n = 0$, then clearly $q = g \cdot p$. Suppose $n \neq 0$. Then for all $\langle h, m\rangle \in G \times \mathbb{Z}$, $q^*(h, m) = p^*(g^{-1}h, m - n)$. In particular, $q(h)(n) = q^*(h, n) = p^*(g^{-1}h, 0) = 2$, a contradiction. ⊣

One of the issues with $E(\mathbb{F}_\omega, 2^{\mathbb{N}})$ is that it's basically impossible to visualize. We want to show $E_\infty = E(\mathbb{F}_2, 2)$ is universal countable Borel instead by some reductions.

4 • 11. Proposition

Let G be a countable group and let $C_2 = \{0, 1\}$ be the cyclic group of order 2. Then
$$E(G, 3) \leqslant_B E(G \times C_2, 2)$$

Proof .:.

Consider the Borel map $f : 3^G \to 2^{G \times C_2}$ where $p \mapsto p^*$ defined by

$$p^*(g,i) = \begin{cases} 0 & \text{if } p(g) = 0 \\ 0 & \text{if } p(g) = 1 \text{ and } i = 0 \\ 1 & \text{if } p(g) = 1 \text{ and } i = 1 \\ 1 & \text{if } p(g) = 2. \end{cases}$$

We claim that f is a Borel reduction from $E(G,3)$ to $E(G \times C_2, 2)$. First suppose $p, q \in 3^G$ and there exists $g \in G$ such that $q = g \cdot p$. Then clearly $q^* = (g,0)p^*$. Next suppose that $q^* = (g,i)p^*$. If $i = 0$, then clearly $q = gp$, as you don't disturb the coding.

So suppose $i = 1$. First consider the case where there exists an $h \in G$ such that $q(h) = 1$. Then $q^*(h,0) = 0$ and $q^*(h,1) = 1$. Note that for all $\langle a, j \rangle \in G \times C_2$,

$$q^*(a,j) = (g,1)p^*(a,j) = p^*(g^{-1}a, j+1).$$

In particular, $p^*(g^{-1}h, 0) = q^*(h,1) = 1$ and so $p(g^{-1}h) = 2$ while $p^*(g^{-1}h, 1) = q^*(h,0) = 0$, which is a contradiction. Thus $q \in \{0,2\}^G$, and it follows that $q = g \cdot p$. $\dashv$

Now we can go through to prove Target Theorem (4 • 4).

Proof of Target Theorem (4 • 4) .:.

Let E be a countable Borel equivalence relation. Then

$$E \leqslant_{\mathrm{B}} E(\mathbb{F}_\omega, 2^{\mathbb{N}}) \equiv_{\mathrm{B}} E(\mathbb{F}_\omega, 2^{\mathbb{Z}\setminus\{0\}} \leqslant_{\mathrm{B}} E(\mathbb{F}_\omega \times \mathbb{Z}, 3) \leqslant_{\mathrm{B}} E(\mathbb{F}_\omega \times \mathbb{Z} \times C_2, 2) \leqslant_{\mathrm{B}} E(\mathbb{F}_\omega, 2)$$

since there is a surjection from $\mathbb{F}_\omega$ to $\mathbb{F}_\omega \times \mathbb{Z} \times C_2$. And this Borel reduces to $E(\mathbb{F}_2, 2)$ since $\mathbb{F}_\omega \hookrightarrow \mathbb{F}_2$.

Recall that by HKL (Harris–Kechris–Louveau?), E_0 is the $<_{\mathrm{B}}$-successor of $\mathrm{id}_{2^{\mathbb{N}}}$. We next study countable Borel equivalence relations E such that $E \equiv_{\mathrm{B}} E_0$.

4 • 12. Definition

A Borel equivalence relation E is *finite* iff every E-class is finite.
A Borel equivalence relation E on a standard Borel space X is *hyperfinite* iff there exists an increasing chain

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on X such that $E = \bigcup_{n\in\omega} F_n$.

Remark: If F is a finite Borel equivalence relation on X, then F is smooth. This is just because X is Borel isomorphic to $\mathbb{R}$: there is a Borel linear order $<$ of X, and so we can define a Borel selector s by taking $s(x) = \min([x]_E)$.

Remark: E_0 is hyperfinite. To see this, define F_n on $2^{\mathbb{N}}$ by $x \; F_n \; y$ iff $x(\ell) = y(\ell)$ for all $\ell \geq n$. Then $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$, and $E_0 = \bigcup_{n\in\omega} F_n$.

4 • 13. Open Problem

Suppose that $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots$ for $n \in \omega$ are hyperfinite Borel equivalence relations. Is $E = \bigcup_{n\in\omega} E_n$ hyperfinite?

Despite looking trivial, this isn't so easy. And Simon himself conjectures that it's probably false. Consider the following characterization of hyperfinite Borel equivalence relations due to Dougherty–Jackson–Kechris.

4 • 14. Theorem

If E is a non-smooth hyperfinite Borel equivalence relation, then $E \equiv_{\mathrm{B}} E_0$.

The new target theorem is the following:

4•15. Theorem (Second Target Theorem)

If E is a countable Borel equivalence relation on a standard Borel space X, then the following are equivalent:

(a) E is hyperfinite.

(b) There exists a Borel action $\mathbb{Z} \curvearrowright X$ such that $E_{\mathbb{Z}}^{X} = E$.

Proof of (a) implies (b) .:.

First we show that (a) implies (b). Express $E = \bigcup_{n\in\omega} F_n$ as an increasing chain $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$ of finite Borel equivalence relations such that $F_0 = \mathrm{id}_X$. Let $<$ be a Borel linear ordering of X. Then we can define an increasing sequence of Borel partial orderings $<_n$ of X as follows:

- $<_0 = \emptyset$.
- $x <_{n+1} y$ iff $x <_n y$ or ($x \; F_{n+1} \; y$ and $\min([x]_{F_n}) < \min([y]_{F_n})$).

Then we see inductively that

1. $<_n$ linearly orders each F_n-class.
2. If $C \neq D$ are F_n-classes such that $[C]_{F_{n+1}} = [D]_{F_{n+1}}$, then either $C <_{n+1} D$ or $D <_{n+1} C$.

Let $< = \bigcup_{n<\omega} <_n$. Then $<_\omega$ is a Borel partial order which linearly orders each E-class. Furthermore, the order type of each E-class is either

(i) n for some $n \geq 1$;

(ii) ω;

(iii) ω^*, the reverse of ω; or

(iv) $\omega^* + \omega$, the order type of $\mathbb{Z}$.

We obtain a $\mathbb{Z}$-action by defining a Borel bijection $T : X \to X$ as follows.

Case i. Suppose $[x]_E$ has order type $1 \leq n < \omega$ under $<_\omega$, say, $x_0 <_\omega x_1 <_\omega \cdots <_\omega x_{n-1}$. Then we define

$$T(x_i) = \begin{cases} x_{i+1} & \text{if } i < n-1 \\ x_0 & \text{if } i = n-1 \end{cases}.$$

Case ii. Suppose that $[x]_E$ has order type ω under $<_\omega$, say, $x_0 <_\omega x_1 <_\omega \cdots <_\omega x_n <_\omega \cdots$. Then T acts on $[x]_E$ as the infinite cycle

$$(\cdots x_3 \, x_1 \, x_0 \, x_2 \, x_4 \cdots)$$

Case iii. If $[x]$ has order type ω^* under $<_\omega$, we handle it similarly to (Case ii).

Case iv. Finally, if $[x]_E$ has order type $\omega^* + \omega$ under $<_\omega$, then we define $T(z)$ to be the $<_\omega$-successor of z.

The proof that (b) implies (a) needs some preparation. First note that if E is a countable Borel equivalence relation on X, then

$$Y = \{x \in X : [x]_E \text{ is finite}\}$$

is Borel and $E \restriction Y$ is finite and hence hyperfinite.

Hence we can restrict our attention to aperiodic equivalence relations, i.e. those equivalence relations such that every class is infinite.

4•16. Definition

Let E be an aperiodic countable Borel equivalence relation. Then a *vanishing sequence of markers* is a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$ such that

1. Each A_n is a complete Borel section for E, i.e. $A_n \cap [x]_E \neq \emptyset$ for all $x \in X$; and
2. $\bigcap_{n\in\omega} A_n = \emptyset$.

4•17. Lemma (The Marker Lemma)

Every aperiodic countable Borel equivalence relation admits a vanishing sequence of markers.

Proof .:.

Without loss of generality, E is a relation on $2^{\mathbb{N}}$. For each $x \in 2^{\mathbb{N}}$, and $n \in \omega$, let $S_n(x)$ be the lexiographic-least $s \in 2^n$ such that $|[x]_E \cap U_s|$ is infinite, where $U_s = \{f \in 2^n : s = f\}$. Then we define $x \in A_n$ iff $x \restriction n = S_n(x)$. Then $\langle A_n : n \in \omega\rangle$ is a decreasing sequence of Borel subsets which intersects each E-class in infinitely many elements. Note that $A = \bigcap_{n\in\omega} A_n$ intersects each E-class in at most one element. Thus $\{A_n \setminus A : n \in \omega\}$ is a vanishing sequence of markers. ⊣

Now we prove that (b) implies (a) from above.

Proof of (b) implies (a) from Second Target Theorem (4 • 15) .:.

We want to show that if $\mathbb{Z} \curvearrowright X$ is a Borel action on a Standard Borel space, then $E = E_{\mathbb{Z}}^X$ is hyperfinite.

So without loss of generality, E is aperiodic. Let $T : X \to X$ be a Borel bijection which generates the $\mathbb{Z}$-action and let $<$ be the Borel partial order on X defined by

$$x < y \quad \text{iff} \quad \exists n > 0(T^n(x) = y).$$

Then $<$ gives a $\mathbb{Z}$-ordering of every E-class. By The Marker Lemma (4 • 17), let $\{A_n : n \in \omega\}$ be a vanishing sequence of markers for E. Define

$$Y = \{x \in X : \text{there is an } n \in \omega \text{ such that } A_n \cap [x]_E \text{ has a } <\text{-least or greatest element}\}.$$

Then Y is an E-invarant Borel subset such that $E \restriction Y$ is smooth, because we have a Borel selector $f : Y \to Y$ for E. Now for each $n \in \omega$, we can define a finite Borel equivalence relation F_n by

$$x \ F_n \ y \quad \text{iff} \quad x = y \vee x, y \in \{T^{\ell}(f(x)) : -n \le \ell \le n\}.$$

Then $F_0 \le F_1 \le \cdots \le F_n \le \cdots$, and $E = \bigcup_{n\in\omega} F_n$. Thus $E \restriction Y$ is hyperfinite.

So without loss of generality, $Y = \emptyset$. Thus for each $x \in X$ and each $n \in \omega$, $A_n \cap [x]_E$ is unbounded in both directions. Hence we can define a finite Borel equivalence relation by

$$x \ F_n \ y \quad \text{iff} \quad x = y \vee A_n \cap [x, y] = \emptyset.$$

Then $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$ and $E = \bigcup_{n\in\omega} F_n$. Thus E is hyperfinite. ⊣

Now we get some closure operations on the class of hyperfinite Borel equivalence relations.

4 • 18. Theorem

Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y.

(a) If $X = Y$, and $E \subseteq F$ with F hyperfinite, then E is hyperfinite (trivial).
(b) If F is hyperfinite and $E \leqslant_{\mathrm{B}} F$, then E is hyperfinite (non-trivial).
(c) If E is hyperfinite and $A \subseteq X$ is Borel, then $E \restriction A$ is hyperfinite (trivial).
(d) If A is a complete Borel section for E and $E \restriction A$ is hyperfinite, then E is hyperfinite (non-trivial).
(e) If E and F are hyperfinite, then $E \times F$ on $X \times Y$ is hyperfinite (trivial)

Proof .:.

(d) By Feldman–Moore Theorem (2 • 2), there exists a countable group $G = \{g_n : n \in \omega\}$ and a Borel action $G \curvearrowright X$ such that $E = E_G^X$. For each $n \in \omega$, let $n(x)$ be the least $n \in \omega$ such that $g_n \cdot x \in A$. Let $E \restriction A = \bigcup_{n\in\omega} F_n$ where $\{F_n : n \in \omega\}$ is an increasing sequence of finite Borel equivalence relations.

Let E_n be the finite Borel equivalence relation such that $x \ E_n \ y$ iff either $x = y$ or $n(x), n(y) < n$ and $g_{n(x)} \cdot x \ F_n \ g_{n(y)} \cdot y$. Clearly, $E = \bigcup_{n\in\omega} E_n$ is hyperfinite.

(b) Let $f : X \to Y$ be a Borel reduction from E to F. Since f is countable-to-one, $A = f"X$ is a Borel subset of Y, and there exists a Borel inverse $g : A \to X$. By (c), $F \restriction A$ is hyperfinite. Also, since g is injective, it follows that $B = g"A$ is Borel, and it is clearly a complete section, since each $x \in X$ is mapped to something in A, which is taken to another representative by g. Since $E \restriction B \cong F \restriction A$, it follows that $E \restriction B$ is hyperfinite. By (d), E is also hyperfinite. ⊣

Recall Open Problem 4 • 13: if $E = \bigcup_{n\in\omega} E_n$ is the union of hyperfinite Borel equivalence relations, is E hyperfinite? Consider a related idea using the domination order.

4 • 19. Definition

If $f, g \in \mathbb{N}^{\mathbb{N}}$, then $f \leqslant^* g$ iff $f(n) \leq g(n)$ for all but finitely many $n \in \mathbb{N}$. Write $f =^* g$ iff $f(n) = g(n)$ for all but finitely many n, so that $f =^+ g$ iff $f \leqslant^* g$ and $g \leqslant^* f$.

Note that $=^*$ is clearly a countable Borel equivalence relation, and it is easily checked that $=^* \equiv_{\mathrm{B}} E_0$. Observe that if $\{f_n : n \in \mathbb{N}\} \subseteq \mathbb{N}^{\mathbb{N}}$, then there exists a $g \in \mathbb{N}^{\mathbb{N}}$ such that $f_n \leqslant^* g$ for all $n \in \omega$. To see this, just define $g(\ell) = 1 + \max_{n\leq\ell} f_n(\ell)$.

Observe also that if E is a countable Borel equivalence relation on a standard Borel space X, and $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ is a Borel map, then there exists a map $\theta : X \to \mathbb{N}^{\mathbb{N}}$ such that

- if $x\ E\ y$, then $\theta(x) = \theta(y)$;
- $\varphi(x) \leq^* \theta(x)$ for all $x \in X$.

Note that we can't ensure that θ is Borel unless E is smooth.

4 • 20. Theorem

There exists a Borel map $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that there doesn't exist a Borel map $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ satisfying

- $x\ E_0\ y$ implies $\theta(x) = \theta(y)$;
- $\varphi(x) \leqslant^* \theta(x)$ for all $x \in 2^{\mathbb{N}}$.

The proof of this will be delayed until the next section. Suppose we have this map φ. When we look at θ, mapping to the Baire space, we are sending E_0 things to identical things. Since E_0 isn't smooth, this shouldn't be a Borel reduction. So this map must have a huge kernel in the sense that we get a counter example to the second property. To show this, we will use category. Why not use measure? Because it cannot work here.

We will make use of one of the only two theorems Simon knows from probability theory: the Borel–Cantelli Lemma below.

4 • 21. Lemma (Borel–Cantelli)

Suppose $\langle X, \mu\rangle$ is a standard Borel probability space, and $E_n \subseteq X$ is a Borel subset for each $n \in \mathbb{N}$. If $\sum_{n\in\mathbb{N}} \mu(E_n) < \infty$, then

$$\mu\left(\{x \in X : x \in E_n \text{ for infinitely many } n \in \mathbb{N}\}\right) = 0.$$

4 • 22. Theorem

Let $\langle X, \mu\rangle$ be a standard Borel probability space. Let $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ be any Borel map. Then there exists a *fixed* $h \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu\left(\{x \in X : \varphi(x) \leqslant^* h\}\right) = 1.$$

Proof .:.

For each $n \in \mathbb{N}$, there exists an $h(n) \in \mathbb{N}$ such that $\mu(\{x \in X : \varphi(x)(n) > h(n)\}) < 2^{-(n+1)}$. By Borel–Cantelli (4 • 21),

$$\mu(\{x \in X : \varphi(x) \leqslant^* h\}) = 1 \dashv$$

4 • 23. Definition

Let E be a countable Borel equivalence relation on a standard Borel space. Then E is *Borel-Bounded* iff for every Borel map $\varphi : X \to \mathbb{N}^{\mathbb{N}}$, there exists a Borel homomorphism $\theta : X \to \mathbb{N}^{\mathbb{N}}$ from E to $=^*$ such that $\varphi(x) \leqslant^* \theta(x)$ for all $x \in X$.

4 • 24. Theorem

If E is hyperfinite, then E is Borel-Bounded.

Proof .:.

Express $E = \bigcup_{n\in\omega} F_n$ as the union of a chain of finite Borel equivalence relations. Define $\theta : X \to \mathbb{N}^{\mathbb{N}}$ by

$$\theta(x)(n) = \max\{\varphi(y)(n) : y\ F_n\ x\}.$$

Then θ is a homomorphism from E to $=^*$, and $\varphi(x)(n) \leq \theta(x)(n)$ for all $n \in \omega$ ⊣

4 • 25. Open Problem

Is every Borel-Bounded countable Borel equivalence relation hyperfinite?
Does there exist a non-Borel-Bounded, countable Borel equivalence relation?

Remark: assuming Martin's Conjecture, $\equiv_{\mathrm{T}}$ isn't Borel-Bounded[ii]. In fact, it's enough to show that if $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_{\mathrm{T}}$ to E_0, then θ sends a cone to a single E_0 class.

Exercise 1

Suppose E and F are countable Borel equivalence relations. If $E \leqslant_{\mathrm{B}} F$ and F is Borel-Bounded, then E is also Borel-Bounded.

Section 5. Baire Category Methods

5 • 1. Definition

Let X be a Polish space.

- $C \subseteq X$ is *comeager* iff there are dense open subsets D_n for $n < \omega$ such that $\bigcap_{n\in\omega} D_n \subseteq C$.
- $M \subseteq X$ is *meager* iff $X \setminus C$ is comeager.

Recall the following theorem.

5 • 2. Theorem (Baire Category Theorem)

If C is a comeager subset of a Polish space, then C is dense in X.

Usually, we only require C to be non-empty.

5 • 3. Definition

Let X be a Polish space. Then $A \subseteq X$ has the *Baire property* (BP) iff there exists an open $U \subseteq X$ such that $A \triangle U$ is meager.

Which sets have the Baire property? The following theorem tells us that we get the Borel sets together with the meager sets, in the sense that the σ-algebra generated by these has the Baire property.

5 • 4. Theorem

Let X be a Polish space and $\mathcal{C}$ be the set of subsets of X with the Baire property. Then $\mathcal{C}$ is the σ-algebra generated by the open sets and the meager sets.

5 • 5. Corollary

If $A \subseteq X$ is Borel, then A has the Baire property.

Using this and the following couple of results, it's very easy to see that E_0 isn't smooth.

5 • 6. Theorem

Suppose X, Y are Polish and $f : X \to Y$ is Borel. Then there exists a comeager $C \subseteq X$ such that $f \restriction C$ is continuous.

[ii] Martin's Conjecture is that the only homomorphisms from $\equiv_{\mathrm{T}}$ to $\equiv_{\mathrm{T}}$ are the jumps on cones.

Proof .:.

Let $\{U_n : n < \omega\}$ be an open basis for the toppology of Y. Since each $f^{-1}"U_n$ is Borel, there exists an open V_n such that $M_n = f^{-1}"U_n \triangle V_n$ is meager. Let $C_n = X \setminus M_n$. Then $C = \bigcap_{n\in\omega} C_n$ is comeager. Also $f^{-1}"U_n \cap C = V_n \cap C$. But this is saying precisely that $f \restriction C$ is continuous. $\dashv$

5•7. Proposition

Suppose that $\Gamma \curvearrowright X$ is a continuous action of a countable group Γ on a Polish space X. If $A \subseteq X$ is comeager, then $\{x \in X : \Gamma \cdot x \subseteq A\}$ is comeager.

Proof .:.

Since $\Gamma \curvearrowright X$ is continuous, for each $g \in \Gamma$, $g^{-1}A$ is also comeager. Hence $B = \bigcap_{g\in\Gamma} g^{-1}A$ is also comeager. If $x \in B$, then $x \in g^{-1}A$ for all $g \in \Gamma$. And so $g \cdot x \in A$ for all $g \in \Gamma$. $\dashv$

We next give a category proof of the fact that $\mathrm{id}_{2^{\mathbb{N}}} <_{\mathrm{B}} E_0$. In fact, we prove the following, stronger result sometimes called "generic ergodicity" theorem.

5•8. Theorem (Generic Ergodicity Theorem)

If X is a Polish space and $\theta : 2^{\mathbb{N}} \to X$ is a Borel homomorphism from E_0 to id_X, then there exists a comeager $C \subseteq 2^{\mathbb{N}}$ such that $\theta \restriction C$ is a constant. In particular, $\mathrm{id}_{2^{\mathbb{N}}} <_{\mathrm{B}} E_0$.

Proof .:.

For each $n \in \omega$, let $\pi_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel bijection defined by

$$(x_0, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots) \xmapsto{\pi_n} (x_0, \cdots, x_{n-1}, 1 - x_n, x_{n+1}, \cdots).$$

Take $\Gamma = \bigoplus_{n\in\omega} \langle \pi_n \rangle$. Then $\Gamma \curvearrowright 2^{\mathbb{N}}$ is continuous and the orbit equivalence relation is E_0. Also notice that for all $x \in 2^{\mathbb{N}}$, the orbit $\Gamma \cdot x$ is dense in $2^{\mathbb{N}}$.

Suppose that $\theta : 2^{\mathbb{N}} \to X$ is a Borel homomorphism from E_0 to id_X: Then there exists a comeager $C \subseteq 2^{\mathbb{N}}$ such that $\theta \restriction C$ is continuous. Since $\Gamma \curvearrowright 2^{\mathbb{N}}$ is continuous, there exists a comeager $D \subseteq C$ such that $\Gamma \cdot x \subseteq C$ for all $x \in D$. So fix some $x \in D$. Then

- θ is constant on $\Gamma \cdot x$.
- $\Gamma \cdot x$ is dense in C.
- θ is continuous on C.

Hence $\theta \restriction C$ is constant. $\dashv$

5•9. Theorem

There exists a Borel map $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that there doesn't exist a Borel map $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that
- If $x \; E_0 \; y$ then $\theta(x) = \theta(y)$;
- $\varphi(x) \leqslant^* \theta(x)$ for all $x \in 2^{\mathbb{N}}$.

Proof .:.

We define $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ as follows. If $x \in 2^{\mathbb{N}}$, then there exists $A \subseteq \mathbb{N}$ such that $x = \chi_A$.

Case 1. If A is finite, let $\varphi(x)$ be the identically zero function.

Case 2. Otherwise let $\{a_n : n \in \omega\}$ be the increasing enumeration of A. Then we define $\varphi(x)(n) = a_n$.

Claim 1

For each $h \in \mathbb{N}^{\mathbb{N}}$, $D_h = \{x \in 2^{\mathbb{N}} : \varphi(x) \not\leqslant^* h\}$ is comeager.

Proof .:.

For each $m \in \omega$,

$$D_m^h = \{x \in 2^{\mathbb{N}} : \exists n \geq m(\varphi(x)(n) > h(n))\}$$

is clearly dense open. Hence $D_h \supseteq \bigcap_{m\in\omega} D_M^h$ is comeager. ⊣

Suppose such a map $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ exists, i.e. a Borel homomorphism from E_0 to $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}$. Then there exists a comeager $C \subseteq 2^{\mathbb{N}}$ and a fixed $h \in \mathbb{N}^{\mathbb{N}}$ such that $\theta(x) = h$ for all $x \in C$. Also, $D_h \cap C$ is comeager; and so there exists an $x \in D_h \cap C$. But then $\varphi(x) \not\leq^* h = \theta(x)$, a contradiction. ⊣

The next target theorem is that category isn't useful for anything else.

5 • 10. Theorem (Third Target Theorem)

If E is a countable Borel equivalence relation on a Polish space X, then there exists a comeager E-invariant Borel subset $C \subseteq X$ such that $E \restriction C$ is hyperfinite.

Fortunately, this is false using measure instead of category. To prove this, we will use the Kuratowski–Ulam theorem.

5 • 11. Definition

ith each $A \subseteq X$, we associate the property $A(x)$ iff $x \in A$, and we write $\forall^* x A(x)$ iff A is comeager. Here '$\forall^*$' is sometimes known as a category quantifier.

In essence, the Kuratowski–Ulam theorem says that category quantifiers commute: $\forall^* x \in X \forall^* y \in Y(A(x,y))$ iff $\forall^* y \in Y \forall^* x \in X(A(x,y))$.

5 • 12. Theorem (Kuratowski–Ulam)

Suppose that X, Y are Polish and $A \subseteq X \times Y$ has the Baire property. Therefore

$$A \text{ is comeager} \quad \text{iff} \quad \forall^* x \in X(A_x \text{ is comeager}) \quad \text{iff} \quad \forall^* y \in Y(A^y \text{ is comeager}).$$

Despite looking rather unimportant, the usefullness of this theorem is immense.

5 • 13. Definition

Let E be a countable Borel equivalence relation on a Polish space X. Then a *cascade* is a sequence

$$X = S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$$

of Borel complete sections for E together with Borel retractions $f_n : S_n \to S_{n+1}$. (Here retraction means $f_n \restriction S_{n+1} = \mathrm{id}_{S_{n+1}}$ and $f_n(x) \; E \; x$ for all $x \in S_n$.)

Given a cascade $\{S_n, f_n : n \in \omega\}$, we can define a sequence of Borel equivalence relations E_n by $x \; E_n \; y$ iff $f_n \circ f_{n-1} \circ \ldots \circ f_0(x) = f_n \circ f_{n-1} \circ \ldots \circ f_0(y)$. Then each E_n is smooth; and $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n \subseteq \cdots \subseteq E$.

5 • 14. Lemma

With the above hypothesis, if each f_n is finite-to-one, then $E_\omega = \bigcup_{n\in\omega} E_n$ is hyperfinite.

So the plan will be to construct a suitable cascade such that for most elements of the space, the union of these equivalence relations is the whole space E.

Proof of Third Target Theorem (5 • 10) .:.

Applying Theorem 3 • 6, let $\{g_n : n \in \omega\}$ be a sequence of Borel bijections. $g_n : X \to X$ with $g_n^2 = 1$ such that $g_0 = \mathrm{id}_X$ and $x \; E \; y$ iff $\exists n(g_n(x) = y)$. Also, fix some Borel linear ordering $<$ of X. For each Borel subset $S \subseteq X$, and each $n \in \omega$, let F_n^S be the Borel equivalence relation on S defined by

$$x \; F_n^S \; y \quad \text{iff} \quad x = y \vee g_n(x) = y.$$

As involutions, this is symmetric. Here each F_n^S-class has at most 2 elements. Hence the set $\Phi_n(S)$ of $<$-minimal elements of each F_n^S-class is also Borel. Also, if S is a complete section, so is $\Phi_n(S)$.

Let $f_n^S : S \to \Phi_n(S)$ be the Borel map which sends each $x \in S$ to the $<$-minimal element of $[x]_{F_n^S}$. The strategy for the rest of the proof is to define for each element of Baire space, there is a a corresponding cascade that uses these ingredients, and then use Kuratowski–Ulam (5 • 12).

For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, we define a cascade $\{S_n^\alpha, f_n^\alpha : n \in \omega\}$ by

- $S_0^\alpha = X$;
- $S_{n+1}^\alpha = \Phi_{\alpha(n)}(S_n^\alpha)$; and
- $f_n^\alpha = f_{\alpha(n)}^{S_n^\alpha}$.

Let $\{E_n^\alpha : n < \omega\}$ be the corresponding sequence of finite Borel equivalence relations and let $E_\omega^\alpha = \bigcup_{n\in\omega} E_n^\alpha \subseteq E$ be the corresponding hyperfinite Borel equivalence relation. The theorem follows from the following series of claims.

Claim 1

There exists an $\alpha \in \mathbb{N}^{\mathbb{N}}$ and a comeager E-invariant Borel $C \subseteq X$ such that $E \restriction C = E_\omega^\alpha \restriction C$.

In fact, we prove the stronger claim below.

Claim 2

$\forall^* \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall x \in X\ ([x]_E = [x]_{E_\omega^\alpha})$.

By Kuratowski–Ulam (5 • 12), it is enough to show the following claim instead.

Claim 3

$\forall x \in X\ \forall^* \alpha \in \mathbb{N}^{\mathbb{N}}\ ([x]_E = [x]_{E_\omega^\alpha})$.

Proof .:.

Fix some $x \in X$. It is enough to show that for each $y \in [x]_E$,

$$A = \{\alpha \in \mathbb{N}^{\mathbb{N}} : y \in [x]_{E_\omega^\alpha} = \bigcup_{n\in\omega} [x]_{E_n^\alpha}\}$$

is dense open, since then the countable intersection is comeager.

Claim 4

A is open.

Proof .:.

Suppose that $\alpha \in A$. Then there exists an $n \in \omega$ such that $y \in [x]_{E_n^\alpha}$. It follows that

$$\mathfrak{N}_{\alpha\restriction n+1} = \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta\restriction n+1 = \alpha\restriction n+1\} \subseteq A.$$

is a neighborhood of α, and so A is indeed open. ⊣

Claim 5

A is dense.

Proof .:.

Fix some basic open $\mathfrak{N}_s$ where $s \in \mathbb{N}^{n+1}$. Consider the "finite cascade" defined by

$$S_0, f_0, S_1, f_1, \cdots, S_n, f_n, S_{n+1}.$$

Where these are the functions determined by the initial values. Let $x' = f_n \circ \cdots \circ f_0(x)$ and $y' = f_n \circ \cdots \circ f_0(y)$. Since $x'\ E\ y'$, there exists a $k \in \omega$ such that $g_k(x') = y'$. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be such that $s \subseteq \alpha$ and $\alpha(n+1) = k$. Then $\alpha \in A$. And so A is also dense. ⊣

⊣

⊣

Section 6. Measure theoretic methods

Question: suppose that G is a countable group and $G \curvearrowright X$ is a Borel action on a standard Borel space. Does the complexity of E_G^X reflect the complexity of G?

To some extent, there is some truth here. For example, if $G = \mathbb{Z}$, we can only get hyperfinite things. Unfortunately, there is an easy counterexample. So let G be *any* countable group, and let $G \curvearrowright G \times [0,1]$ be the Borel action $g \cdot \langle h, r\rangle = \langle gh, r\rangle$. Then the Borel map $\langle g, r\rangle \mapsto \langle 1_G, r\rangle$ shows that $E_G^{G\times[0,1]}$ is smooth.

So what is wrong with this, and how can we eliminate this kind of triviality? One issue with the above idea is that the action is free, but also there is no invariant probability measure.

6 • 1. Proposition

Suppose G is a countable group and $G \curvearrowright X$ is a *free* Borel action on a standard Borel space X. If there exists a G-invariant, Borel, probability measure μ on X, then E_G^X isn't smooth.

Proof .:.

Suppose that E_G^X is smooth. Then there exists a Borel transversal $T \subseteq X$. Since $G \curvearrowright X$ is free, $X = \bigsqcup_{g\in G} g"T$. So $1 = \sum_{g\in G} \mu(g"T) = \sum_{g\in G} \mu(T)$, which is a contradiction. ⊣

6 • 2. Definition

Let G be countable and let X be a standard Borel G-space. A G-invariant probability measure μ is *ergodic* iff whenever $A \subseteq X$ is a G-invariant Borel subset, then $\mu(A) = 1$ or $\mu(A) = 0$.

We have another characterization due to the following theorem. Note that $f : X \to Y$ is G-invariant iff $f(g\cdot x) = f(x)$ for all $x \in X$ and $g \in G$.

6 • 3. Theorem

If μ is a G-invariant, Bore, probability measure, then the following are equivalent:

1. μ is ergodic;
2. If Y is a standard Borel space, and $f : X \to Y$ is G-invariant, then there is a G-invariant Borel $M \subseteq X$ with $\mu(M) = 1$ and such that $f \restriction M$ is constant.

Proof .:.

(2) → (1). Suppose $A \subseteq X$ is a G-invariant, Borel subset. Define $f : X \to 2$ by $f(x) = 1$ iff $x \in A$. Since A is G-invariant, f is G-invariant. Hence there exists a G-invariant Borel $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is constant. So $\mu(A) = 1$ or $\mu(A) = 0$.

(1) → (2). Without loss of generality, $Y = [0,1]$, since we can expand to an uncountable space if necessary, and it will be isomorphic to $[0,1]$. Let $Z_0 = [0,1]$. Suppose inductively that we have defined an interval

$$Z_n = \left[\frac{a_n}{2^{n-1}}, \frac{a_n+1}{2^{n-1}}\right]$$

for some $0 \le a_n < 2^{n-1}$ such that $\mu(f^{-1}"Z_n) = 1$. Let

$$I_{n+1} = \left[\frac{2a_n}{2^n}, \frac{2a_n+1}{2^n}\right] \qquad J_{n+1} = \left(\frac{2a_n+1}{2^n}, \frac{2a_n+2}{2^n}\right].$$

Then $f^{-1}"I_{n+1}$ and $f^{-1}"J_{n+1}$ are G-invariant, Borel subsets. And so either $\mu(f^{-1}"I_{n+1}) = 1$ or $\mu(f^{-1}"J_{n+1}) = 1$. In the former case, let $Z_{n+1} = I_{n+1}$; in the latter, set $Z_{n+1} = \overline{J_{n+1}}$, the closure of J_{n+1}.

Clearly, $M = \bigcap_{n\in\omega} f^{-1}\text{"}Z_n$ is a G-invariant, Borel subset with $\mu(M) = 1$, and such that $f \restriction M$ is constant. ⊣

Now we want something stronger than merely being ergodic.

6•4. Definition

Let G be a countable group, and let X be a standard Borel G-space. The action $G \curvearrowright X$ is *uniquely ergodic* iff there is a unique G-invariant probability measure μ on X.

Clearly, we should have the following.

6•5. Theorem

If $G \curvearrowright \langle X, \mu\rangle$ is uniquely ergodic, then $G \curvearrowright \langle X, \mu\rangle$ is ergodic.

Proof .:.

Suppose that $G \curvearrowright \langle X, \mu\rangle$ isn't ergodic. We need to find two different probability measures. Then there exists a Borel $A \subseteq X$ such that $0 < \mu(A) < 1$, and $0 < \mu(X \setminus A) < 1$ with both A and $X \setminus A$ G-invariant. Then we can define G-invariant probability measures $\nu_1 \neq \nu_2$ by

$$\nu_1(Y) = \frac{\mu(Y \cap A)}{\mu(A)}$$

$$\nu_2(Y) = \frac{\mu(Y \cap (X \setminus A))}{\mu(X \setminus A)} \dashv$$

6•6. Example (Abstract Example)

Let K be a separable, compact group. Then there exists a unique probability measure μ on K such that μ is K-invariant under the left-translation action of $K \curvearrowright K$; namely, the Haar measure. If $\Gamma < K$ is a countable, dense subgroup, then $\Gamma \curvearrowright \langle K, \mu\rangle$ is uniquely ergodic.

6•7. Example (Concrete Example)

Let $K = \prod_{n\in\omega} C_n$ where each $C_n = \{0, 1\}$ of order 2. Then K is compact and the Haar measure is the usual uniform product probability measure. Let $\Gamma = \bigoplus_{n\in\omega} C_n$, which is a dense subgroup. Then $E_\Gamma^K = E_0$ and $\Gamma \curvearrowright \langle K, \mu\rangle$ is uniquely ergodic.

This then gives a theorem about E_0, which is a third proof that E_0 isn't smooth. In some sense, this states that the kernel is large in terms of measure, whereas last time we proved that the kernel is large in terms of category.

6•8. Theorem

Let μ be the usual product measure on $2^{\mathbb{N}}$. If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from E_0 to $\mathrm{id}_{2^{\mathbb{N}}}$, then there exists an E_0-invariant Borel subset $M \subseteq 2^{\mathbb{N}}$ with $\mu(M) = 1$ such that $f \restriction M$ is constant.

We often want every infinite subgroup $H < G$ to act ergodically. Here's a non-example of this happening.

6•9. Example (Non-example)

Let $K = \prod_{n\in\omega} C_n$ be as above. Let $\Delta = \bigoplus_{n\geq 1} C_n$ and $H = \prod_{n\geq 1} C_n$. Then H is Δ-invariant and $\mu(H) = \frac{1}{2}$, since it's an index 2-subgroup.

6 • 10. Definition

Let G be a countable group and let X be a standard Borel G-space with invariant probability measure μ. Then $G \curvearrowright \langle X, \mu \rangle$ is *strongly mixing* iff for any Borel subsets $A, B \subseteq X$, and any sequence $\langle g_n : n \in \omega \rangle$ of distinct elements of G, either $\mu(B) = 0$ or

$$\lim_{n\to\infty} \frac{\mu(g_n"A \cap B)}{\mu(B)} = \mu(A).$$

Note that if $\mu(B) = 0$ we still have $\lim_{n\to\infty} \mu(g"A \cap B) = 0 = \mu(B) \cdot \mu(A)$. Note further that the literature often has "weakly mixing" in addition to "strongly mixing", so one must be careful with which version is referred to if just "mixing" is used.

With the above hypotheses, if $H \leqslant G$ is an infinite subgroup, then $H \curvearrowright \langle X, \mu \rangle$ is also strongly mixing. It also follows that if you're strongly mixing, then you're ergodic, just by taking $A = B$ for a G-invariant A.

6 • 11. Theorem

If $G \curvearrowright \langle X, \mu \rangle$ is strongly mixing, then $G \curvearrowright \langle X, \mu \rangle$ is ergodic.

Proof .:.

Suppose $A \subseteq X$ is Borel and G-invariant. Let $G = \{g_n : n \in \omega\}$. Thus

$$\mu(A)^2 = \lim_{n\to\infty} \mu(g_n"A \cap A) = \lim_{n\to\infty} \mu(A) = \mu(A).$$

This implies $\mu(a)$ is either 0 or 1. ⊣.

As a convention, if G is countably infinite, then the uniform product probability measure on 2^G is denoted by μ. Clearly $G \curvearrowright \langle 2^G, \mu \rangle$ is measure-preserving. This is usually called the Bernoulli action.

6 • 12. Theorem

$G \curvearrowright \langle 2^G, \mu \rangle$ is strongly mixing.

Proof .:.

First suppose that there exist finite $S, T \subseteq G$ and subsets $\mathcal{F} \subseteq 2^S$, $\mathcal{H} \subseteq 2^T$ such that

$$A = \{f \in 2^G : f \restriction S \in \mathcal{F}\} \qquad B = \{f \in 2^G : f \restriction T \in \mathcal{H}\}.$$

Suppose $\langle g_n : n \in \omega \rangle$ is a sequence of distinct elements of G. Since $G \curvearrowright G$ is free, for all but finitely many $n \in \omega$, $g_n(S) \cap T = \emptyset$; and so the events $g_n(A)$ and B are independent:

$$\frac{\mu(g_n"A \cap B)}{\mu(B)} = \mu(g_n"A) = \mu(A).$$

Thus the limit $\lim_{n\to\infty} \mu(g_n"A \cap B) = \mu(A)\mu(B)$. In general, if $C \subseteq 2^G$ is Borel, then for every $\varepsilon > 0$, there exists a finite $S \subseteq G$ and an $\mathcal{F} \subseteq 2^S$ such that

$$\mu\left(C \mathbin{\triangle} \left\{f \in 2^G : f \restriction S \in \mathcal{F}\right\}\right) < \varepsilon.$$

The result follows easily. ⊣

6 • 13. Definition

If G is countably infinite, then

$$(2)^G = \{x \in 2^G : \forall g \in G \setminus \{1_G\}(g \cdot x \neq x)\}$$

is the free part of $G \curvearrowright 2^G$. We define $E_G = E_G^{(2)^G}$.

Fortunately, we keep the probability measure around.

6 • 14. Proposition

$\mu((2)^G) = 1$, where μ is the uniform product probability measure on 2^G.

Proof .:.

It is enough to show that for each $1 \neq g \in G$,
$$\mu(\{x \in 2^G : g \cdot x = x\}) = 0.$$
Let $H = \langle g \rangle$ be the subgroup generated by g. Then $g \cdot x = x$ iff x is constant on each coset Ht. The result follows easily. ⊣

Note that if $H \leqslant G$, then $E_H \leqslant_B E_G$. To see this, we can define a Borel reduction $x \mapsto x^*$ by
$$x^*(g) = \begin{cases} x(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$
Here's a question: to what extent does the converse hold? The answer is that it doesn't hold in general, since it's obviously false for hyperfinite things, but we will move to a setting where the it is almost true. In particular, we will move to the world where we can apply Popa superrigidity, and in particular, groups with a normal, Kazhdan group.

6 • 15. Definition

A countable Borel equivalence relation E on a standard Borel space X is *free* iff there is a free Borel action $G \curvearrowright X$ of a countable group such that $E = E_G^X$.

For example, E_G is clearly free.

6 • 16. Definition

E is *essentially free* if there exists a free countably Borel equivalence relation F such that $E \leqslant_B F$.

For example, take $(2)^{\mathbb{Z}} \sqcup \{x_0\}$, since we can just add a few elements to make sure the action on x_0 is free. A question is now brought up: is everything essentially free? In particular, is E_∞ essentially free? This was answered by Simon using an easy consequence of Popa Superrigidity. First some definitions.

6 • 17. Definition

Let E be a countable Borel equivalence relation on a standard Borel space X. A probability measure μ on X is *E-invariant* if for some (equivalently every) Borel action $G \curvearrowright X$ of a countable group such that $E = E_G^X$, μ is G-invariant.

6 • 18. Definition

Suppose E, F are countable Borel equivalence relation on X, Y, and μ is an E-invariant probability measure. Then a Borel homomorphism $f : X \to Y$ from E to F is *μ-tivial* iff there exists a Borel $Z \subseteq X$ with $\mu(Z) = 1$ such that f sends Z to a single F-class.

6 • 19. Definition

If G, H are countable groups, then a homomorphism $\pi : G \to H$ is a *virtual embedding* iff $|\ker \pi| < \aleph_0$.

The following is an easy consequence of Popa Superrigidity.

6 • 20. Theorem (Black Box)

Let S be *any* countable group and let $G = \mathrm{SL}_3(\mathbb{Z}) \times S$.
Let H be *any* countable group and let $H \curvearrowright Y$ be a free Borel action on a standard Borel space Y.
Therefore, if there exists a μ-nontrivial Borel homomorphism from E_G to E_H^Y, then there exists a virtual embedding from G to H.

Now we give a theorem that shows how we use this, as we are not yet prepared to prove it yet.

6 • 21. Theorem

If E is an essentially free, countable Borel equivalence relation, then there exists a countable G such that $E_G \not\leqslant_B E$.

6 • 22. Corollary

E_∞ isn't essentially free.

In fact, this says that of the essentially free Borel equivalence relations, there isn't a universal one. To prove Theorem 6 • 21, we will make use of two group-theoretic results.

6 • 23. Theorem (B.H. Neumann)

There exist uncountably many finitely generated groups up to isomorphism.

6 • 24. Proposition

If L is any group, then the free product $L * \mathbb{Z}$ has no nontrivial, finite, normal subgroups.

Proof of Theorem 6 • 21 .:.

Without loss of generality, we can suppose that $E = E_H^Y$ for some free Borel action $H \curvearrowright Y$. Since there are uncountably many finitely generated groups. Up to isomorphism, there exists a finitely generated L such that L doesn't embed in H, since H has only countably many finitely generated subgroups. Let $S = L * \mathbb{Z}$. Then S has no nontrivial finite normal subgroups. Finally, let $G = \mathrm{SL}_3(\mathbb{Z}) \times S$.

Suppose $f : (2)^G \to E_H^Y$ is a Borel reudction from E_G to $E_H^Y = E$. Then f is a μ-nontrivial Borel homomorphism. Hence by Black Box (6 • 20), there exists a virtual embedding $\pi : G \to H$. Since $\ker \pi = 1$, it follows that S embeds in H; and hence L embeds in G, a contradiction. ⊣

6 • 25. Theorem

There exist uncountably many (continuum) free, countably Borel equivalence relations up to Borel bireducibility.

In fact, we have a rich supply of non-essentially free ones.

6 • 26. Theorem

There exist uncountably many non-essentially-free countable Borel equivalence relations up to Borel bireducibility.

First we prove Theorem 6 • 25. What examples of essentially free things do we know? So far, E_G. One issue we encounter is to make sure our coding S from Black Box (6 • 20) doesn't embed in $\mathrm{SL}_3(\mathbb{Z})$. To proceed, we will make use of the following theorem

6 • 27. Theorem

$\mathrm{SL}_3(\mathbb{Z})$ contains a torsion-free subgroup of finite index.

Proof .:.

To quote a famous theorem, by Selberg's theorem, every finitely generated, linear group over a field of characteristic 0 contains a torsion-free subgroup of finite index. ⊣

One last thing we require (which it just so happens is something we can prove) is the following.

6 • 28. Lemma

If $H, K \leqslant G$ are any groups, then $[K : K \cap H] \leqslant [G : H]$.

Proof .:.

Let $\{t_i : i \in I\}$ be coset representatives for $K \cap H$ in K. It is enough to show that if $i \neq j$, then $Ht_i \neq Ht_j$. Suppose $Ht_i = Ht_j$. Then there exist $a, b \in H$ such that $at_i = bt_j$ and so $t_i t_j^{-1} = a^{-1}b \in H \cap K$. So that $(H \cap K)t_i = (H \cap K)t_j$, a contradiction. ⊣

Proof of Theorem 6 • 25 .:.

Let $\mathbb{P}$ be the set of primes. For each $p \in \mathbb{P}$, let $A_p = \bigoplus_{n \in \omega} C_p$ be the direct sum of countably many copies of the cyclic group C_p of order p.

For each $C \subseteq \mathbb{P}$, let $G_C = \mathrm{SL}_3(\mathbb{Z}) \oplus \bigoplus_{p \in C} A_p$. We will show that $E_{G_C} \leqslant_{\mathrm{B}} E_{G_D}$ iff $C \subseteq D$, which is certainly enough to get uncountably many of these.

If $C \subseteq D$, then $G_C \leqslant G_D$, and so $E_{G_C} \leqslant_B E_{G_D}$. Conversely, suppose $E_{G_C} \leqslant_B E_{G_D}$. Then there exists a virtual embedding (by Black Box (6 • 20))

$$\pi : \mathrm{SL}_3(\mathbb{Z}) \times \bigoplus_{p \in C} A_p \to \mathrm{SL}_3(\mathbb{Z}) \times \bigoplus_{q \in D} A_q.$$

To make sure the coding actually works, let $N \trianglelefteq \mathrm{SL}_3(\mathbb{Z})$ be a torsion-free subgroup of finite index, and let $F = \mathrm{SL}_3(\mathbb{Z})/N$, a certain finite group. Let

$$\varphi : \mathrm{SL}_3(\mathbb{Z}) \times \bigoplus_{q \in D} A_q \to F \times \bigoplus_{q \in D} A_q$$

be the canonical surjection. The kernel of this will be N, and so it will be torsion free. Fix some $p \in C$, and let $B_p = \pi"A_p \leqslant \mathrm{SL}_3(\mathbb{Z}) \times \bigoplus_{q \in D} A_q$. Then there exists a possibly trivial finite subgroup $N_p \trianglelefteq A_p$ such that $B_p \cong A_p/N_p \cong A_p$. Also note that $B_p \cap \ker\varphi$ is trivial, and hence $E_p = \varphi"B_p \cong A_p$. Finally, note that

$$\left[E_p : E_p \cap \bigoplus_{q \in D} A_q\right] \leq \left[F \times \bigoplus_{q \in D} A_q : \bigoplus_{q \in D} A_q\right] = |F| < \aleph_0$$

Thus $E_p \cap \bigoplus_{q \in D} A_q \cong A_p$. In particular, there is an element of order p, meaning $p \in D$ and thus $C \subseteq D$. ⊣

Now we want to show Theorem 6 • 26. Doing this isn't so simple as with Theorem 6 • 25. The proof of it makes use of the following group theoretic concept and fact.

6 • 29. Definition

The groups G, H are *isomorphic up to finite kernels* iff there exist finite normal subgroups $N \trianglelefteq G$, $M \trianglelefteq H$ such that $G/N \cong H/M$.

6 • 30. Theorem (Group Theoretic Fact)

There exists a Borel family $\{S_x : x \in 2^{\mathbb{N}}\}$ of finitely generated groups such that if $G_x = \mathrm{SL}_3(\mathbb{Z}) \times S_x$, then for all $x \neq y$,

(i) G_x, G_y are not isomorphic up to finite kernels; and
(ii) G_x doesn't virtually embed in G_y.

(i) is only used in showing the non-essentially free part, and (ii) is only used in showing the Borel bireducibility part.

Proof of Theorem 6 • 26 .:.

Using the G_xs from Group Theoretic Fact (6 • 30), for each Borel subset $A \subseteq 2^{\mathbb{N}}$, let $E_A = \bigsqcup_{x \in A} E_{G_x}$ be the smooth disjoint union of $\{E_{G_x} : x \in A\}$, i.e. E_A is the countable Borel equivalence relation on the standard Borel space $X_A = \{\langle x, f\rangle : x \in A, f \in (2)^{G_x}\}$ defined by $\langle x, f\rangle \; E_A \; \langle x', f'\rangle$ iff $x = x' \wedge f \; E_{G_x} \; f'$.

Claim 1

If $A \subseteq 2^{\mathbb{N}}$ is an uncountable Borel subset, then E_A isn't essentially free.

Proof .:.

Suppose $E_A \leqslant_B E_H^Y$, where $G \curvearrowright Y$ is a free Borel action of a countable group H. Then for each $x \in A$, $E_{G_x} \leqslant_B E_H^Y$, just by restricting to each piece. This allows us to use Black Box (6 • 20), and so there exists a virtual embedding $\pi_x : G_x \to H$. Since A is uncountable, and each G_x is finitely generated (H has only countably many finitely generated subgroups), there exist $x \neq y$ such that $\pi_x(G_x) = \pi_y(G_y)$, contradicting that G_x, G_y aren't isomorphic up to finite kernels. ⊣

Claim 2

$E_A \leqslant_B E_B$ iff $A \subseteq B$.

Proof .:.

Clearly if $A \subseteq B$ then $E_A \leqslant_B E_B$. So suppose $E_A \leqslant_B E_B$ and there exists an $x \in A \setminus B$. Then there exists a Borel reduction $f : (2)^{G_x} \to \bigsqcup_{y \in B} (2)^{G_y}$ from E_{G_x} to E_B. Since $G_x \curvearrowright ((2)^{G_x}, \mu_x)$ is ergodic, there exists $Z \subseteq (2)^{G_x}$ with $\mu_x(Z) = 1$ such that $f''Z \subseteq (2)^{G_y}$ for some fixed $y \in B$. But then $f \restriction Z$ yields a μ_x-nontrivial Borel homomorphism from E_{G_x} to E_{G_y}. (Everything outside of Z, we can send to a single element.) And so by Black Box (6 • 20), G_x virtually embeds in G_y, which is a contradiction ⊣

⊣

Finally we sketch the proof of Group Theoretic Fact (6 • 30).

Proof sketch of Group Theoretic Fact (6 • 30) .:.

6 • 31. Definition

An infinite group G is *quasi-finite* iff every proper subgroup of G is finite.

6 • 32. Theorem (Ol'shanskii)

Let $\mathcal{P}$ be the standard Borel space of strictly increasing sequences of primes $x = \langle p_n : n \in \omega \rangle$ such that $p_0 > 10^{75}$. Then there exixts a Borel family $\{T_x : x \in\}$ of 2-generator groups such that for each $x = \langle p_n : n \in \omega \rangle$

- T_x contains a cyclic subgroup of order p_n for each $n \in \omega$.
- Every proper subgroup of T_x is cyclic of order p_n for some $n \in \omega$.
- T_x is simple.

Provint the above theorem takes a huge amount of work, as Ol'shanskii's book shows. Using this result, though, Group Theoretic Fact (6 • 30) follows from the following.

6 • 33. Proposition

For each $x \in \mathcal{P}$, let $G_x = \mathrm{SL}_3(\mathbb{Z}) \times T_x$. Then the Borel family $\{G_x : x \in \mathcal{P}\}$ satisfies that if $x \neq y$, then

- G_x, G_y are not isomorphic up to finite kernels; and
- G_x doesn't virtually embed in G_y.

Proof .:.

Since each T_x is simple and $\mathrm{SL}_3(\mathbb{Z})$ has no nontrivial, finite, normal subgroups; it follows that G_x has no nontrivial, finite, normal subgroups. Thus it is enough to show that if $x \neq y$, then G_x doesn't embed in G_y. Suppose that $\pi : G_x \to G_y$ is an embedding. Certainly π isn't an isomorphism, since we have different primes. In particular, $\pi''G_x$ is an infinite, proper subgroup of G_y, which contradicts Ol'shanskii (6 • 32), as they are all quasi-finite. ⊣

§ 6 A. The Black Box

Now we should attempt to understand why Black Box (6 • 20) is true. To do this, we will need to introduce cocycles. Until further notice, let $G \curvearrowright \langle X, \mu \rangle$ be a Borel action of a countable group G on a standard Borel space X with G-invariant probability measure μ.

6 A • 1. Definition

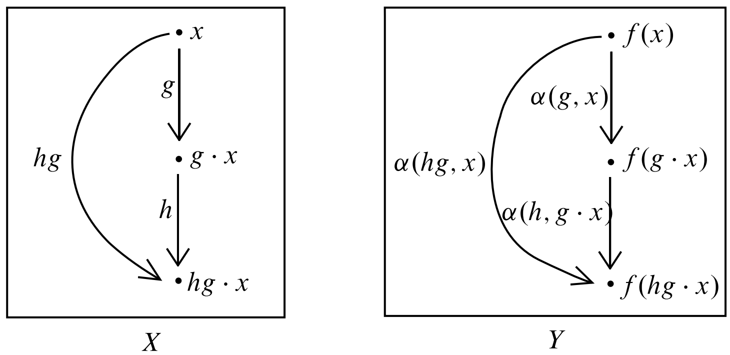
If H is a countable group, then a Borel map $\alpha : G \times X \to H$ is a *cocycle* iff for all $g, h \in G$, $\alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x)$ for μ-almost every $x \in X$.

These are somewhat disgusting, but the theme will be to make these less disgusting, and Popa says you can.

To explain where these come from, cocycles arise naturally as follows. Suppose $H \curvearrowright Y$ is a free Borel action of a countable group H and that $f : X \to Y$ is a Borel homomorphism from E_G^X to E_H^Y. Then we can define a Borel cocycle $\alpha : G \times X \to H$ by

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

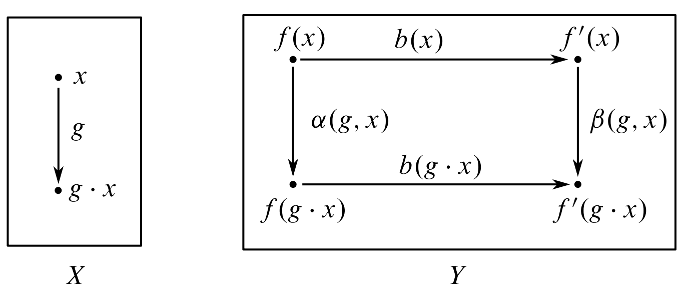
Since $H \curvearrowright Y$ is free, this is how we get uniqueness, as per the following diagram: $f(x)$ must be in the same orbit as $f(gx)$ via $\alpha(g, x)$. $f(gx)$ must be in the same orbit as $f(hgx)$ iva $\alpha(h, gx)$. And so $f(x)$ must be in the same orbit as $f(hgx)$ via $\alpha(hg, x)$.



We are lucky that the following holds. If $\alpha(g, x) = \alpha(g)$ depends only on the g variable, then the cocycle identity reduces to $\alpha(hg) = \alpha(h)\alpha(g)$, meaning $\alpha : G \to H$ is a group homomorphism. Also, in this case using the same setup as the above example, if we look at $(G, X) \to (H, Y)$ using α and f, we get that $\alpha(g)f(x) = f(gx)$ is a homomorphism of permutation groups. So if we can take a function and perturb it to get a function of one variable, then we are in good shape: not only is there a group homomorphism in play, but there's one that respects the group actions. So we'll try to eliminate variables.

Quesetion: can we "adjust" $f : X \to Y$ so that α becomes a group homomorphism? (And what does this "adjusting" mean?)

Suppose that $b : X \to H$ is a Borel map. Then we can define $f' : X \to Y$ by $f'(x) = b(x)f(x) \in Hf(x)$. Then f' is also a Borel homomorphism from E_G^X to E_H^Y (we haven't changed the orbit). If f is a Borel reduction, then so is f'. Similarly, if f is μ-nontrivial, then so is f'. Let $\beta : G \times X \to H$ be the Borel cocycle corresponding to f'.



Thus $\beta(g, x) = b(gx)\alpha(g, x)b(x)^{-1}$. This motivates the following definition.

6 A • 2. Definition

The cocycles $\alpha, \beta : G \times X \to H$ are *equivalent*, written $\alpha \approx \beta$, iff there exists a borel $b : X \to H$ such that for all $g \in G$,

$$\beta(g, x) = b(gx)\alpha(g, x)b(x)^{-1}$$

for μ-almost every $x \in X$.

6 A • 3. Theorem (Popa Superrigidity)

Let Γ be a countable infinite Kazhdan group (defined later) and let G, K be countable groups such that $\Gamma \trianglelefteq G \leqslant K$. Therefore, If H is any countable group, then every Borel cocycle $\alpha : G \times (2)^K \to H$ is equivalent to a group homomorphism from G into H.

In most applications, $G = K$, and so we have a cocycle $\alpha : G \times (2)^G \to H$ and the corresponding orbit equivalence relation is E_G. In many (not all, e.g. Black Box (6 • 20)) applications, $\Gamma = G$ as well.

6 A • 4. Definition

Let Γ be a countable group. Then Γ is a *Kazhdan group* iff there exists a finite subset $F \subseteq \Gamma$ and $\varepsilon > 0$ such that the following holds.

(*) If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is any unitary representation such that there is a unit vector $v \in \mathcal{H}$ with $\|\pi(\gamma)v - v\| < \varepsilon$ for all $\gamma \in F$,

then there exists a Γ-invariant unit vector $v \in \mathcal{H}$.

Note that every countable Kazhdan group is finitely generated, and we can let F be *any* finitely generating set. Furthermore, any homomorphic image of a Kazhdan group is Kazhdan.

6 A • 5. Example

$\mathrm{SL}_n(\mathbb{Z})$ is Kazhdan for all $n \geq 3$.

Recall Black Box (6 • 20), restated below. We now are in a position to prove it.

6 A • 6. Theorem (The Black Box)

Let S be any countable group and let $G = \mathrm{SL}_3(\mathbb{Z}) \times S$. Let H be any countable group and let $H \curvearrowright Y$ be a *free* Borel action. If there exists a μ-nontrivial Borem homomorphism $f : X \to Y$ from E_G to E_H^Y, then there exists a virtual embedding $\pi : G \to H$.

Proof .:.

Suppose that $f : X \to Y$ is a μ-nontrivial Borel homomorphism from E_G to E_H^Y. Then we can define a Borel cocycle $\alpha : G \times (2)^G \to H$ by taking $\alpha(g, x)$ to be the unique $h \in H$ such that $hf(x) = f(hx)$. By Popa Superrigidity (6 A • 3), there exists a Borel map $b : (2)^G \to H$, a group homomorphism $\varphi : G \to H$ and a subset $X \subseteq (2)^G$ with $\mu(X) = 1$ such that for all $g \in G$, $\varphi(g) = b(gx)\alpha(g, x)b(x)^{-1}$ for all $x \in X$.

So let $f' : X \to Y$ be the borel map $f'(x) = b(x)f(x)$. Then f' is also a μ-nontrivial Borel homomorphism from E_G to E_H^Y. Also for all $g \in G$ and $x \in X$,

$$f'(gx) = b(gx)f(gx) = b(gx)\alpha(g,x)f(x) = b(gx)\alpha(g,x)b(x)^{-1}f'(x) = \varphi(g)f'(x). \qquad (**)$$

To see that φ is a virtual embedding, suppose that $N = \ker\varphi$ is infinite. By (**), for all $g \in N$ and $x \in X$, $f'(gx) = f'(x)$. Thus $f' : X \to Y$ is N-invariant. Since $G \curvearrowright \langle X, \mu\rangle$ is strongly mixing, $N \curvearrowright \langle X, \mu\rangle$ is ergodic; and so there exists a $Z \subseteq X$ with $\mu(Z) = 1$ such that f' sends Z to a single point $y_0 \in Y$. But then f sends Z into the single E_H^Y class containing y_0 (we've only adjusted within the same class), which contradicts the fact that f is μ-nontrivial. ⊣

The next application uses the following definition.

6 A • 7. Definition

Let E be a Borel equivalence relation on a standard Borel space X and let $1 \leq n \leq \omega$. Then $nE = E \oplus \cdots \oplus E$ (n times) is the Borel equivalence relation on $X \times n$ defined by

$$\langle i, x\rangle \; nE \; \langle j, y\rangle \quad \text{iff} \quad i = j \wedge x \; E \; y$$

Clearly $nE \leq_B mE$ for $n \leq m$. But it's difficult to show that $E = 1E <_B 2E$ is possible.

6 A • 8. Theorem

There exists a countable Borel equivalence relation E such that
$$E <_{\mathrm{B}} E \oplus E <_{\mathrm{B}} E \oplus E \oplus E <_{\mathrm{B}} \cdots <_{\mathrm{B}} nE <_{\mathrm{B}} \cdots <_{\mathrm{B}} \omega E.$$

Proof .:.

We require two more facts.

Fact 1a. $\mathrm{SL}_3(\mathbb{Z})$ has no nontrivial, finite, normal subgroups.

Fact 1b. If $\pi : \mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z})$ is an injective homomorphism, then π is surjective; i.e. $\mathrm{SL}_3(\mathbb{Z})$ is coHopfian.

As a corollary, if $\pi : \mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z})$ is a virtual embedding, then π is an automorphism.

Fact 2. Suppose G is a countable group and $\langle X, \mu\rangle$ is a standard Borel G-space with invariant probability measure μ. If μ is ergodic, then there exists a G-invariant, Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $G \curvearrowright X_0$ is uniquely ergodic (with unique invariant probability measure $\mu \restriction X_0$).

So let $G = \mathrm{SL}_3(\mathbb{Z})$ and let $X \subseteq (2)^G$ be a G-invariant Borel subset with $\mu(X) = 1$ such that $G \curvearrowright \langle X, \mu\rangle$ is uniquely ergodic. Let $E = E_G^X$. Then clearly $E \leqslant_{\mathrm{B}} E \oplus E \leqslant_{\mathrm{B}} \cdots$. So we want to show that there are no reductions holding in the opposite direction.

Claim 1

If $f : X \to X$ is a μ-nontrivial Borel homomorphism from E to E, then $\mu(G \cdot f"X) = 1$.

Assuming the claim, suppose $\varphi : X \times (n+1) \to X \times n$ is a Borel reduction from $(n+1)E$ to nE. For each $0 \leq i \leq n$, let $X_i = X \times \{i\}$; and for each $0 \leq j \leq n$, let $\varphi_j = \varphi \restriction X_j$. Since $G \curvearrowright \langle X, \mu\rangle$ ergodically, and each is isomorphic to $\langle X, \mu\rangle$, for each $0 \leq j \leq n$, there exists $0 \leq k_j \leq n-1$ and $Z_j \subseteq X_j$; with measure $\mu(Z_j) = 1$ such that $\varphi_n"Z_j \subseteq X_{k_j}$. By the claim, $\mu(G \cdot f"Z_j) = 1$.

There exist by the pigeon-hole principle $i \neq j$ with $k_i = k_j$. But then
$$\mu(G \cdot f"Z_i \cap G \cdot f"Z_j) = 1,$$
and so $G \cdot f"Z_i \cap G \cdot f"Z_j \neq \emptyset$, which is a contradiction: elements which are non-equivalent going into the same class. ⊣

So all that remains to show is the claim. To do this, we make use of some basic observations in measure theory.

6 A • 9. Observation

If $\langle X, \mu\rangle$ is a standard Borel probability space and $f : X \to Y$ is a Borel map, then we can define a probability measure $\nu = f * \mu$ defined by $\nu(A) = \mu(f^{-1}"A)$.

6 A • 10. Observation

With the above hypotheses, suppose $G \curvearrowright \langle X, \mu\rangle$ is a measure-preserving Borel action, $H \curvearrowright Y$ is a Borel action and $\varphi : G \to H$ is a group homomorphism such that $\varphi(g)f(x) = f(gx)$. Then $\nu = f * \mu$ is $\varphi(G)$-invariant.

Proof .:.

Let $A \subseteq Y$ be Borel and $g \in G$. Let $B = f^{-1}"A$. Then $f(gB) = \varphi(g)f(B) = \varphi(g)A$. Hence $\nu(\varphi(g)A) = \mu(gB)$. Since μ is G-invariant, $\mu(gB) = \mu(B) = \nu(A)$. ⊣

Using these observations with Popa Superrigidity (6 A • 3) gives this result easily.

6 A • 11. Result

With X, G, and E as in the proof of Theorem 6 A • 8, if $f : X \to X$ is a μ-nontrivial Borel homomorphism from E to E, then $\mu(G \cdot f"X) = 1$.

Proof .:.

Suppose $f : X \to X$ is a μ-nontrivial Borel homomorphism from E to E. Then we can define a Borel cocycle $\alpha : G \times X \to G$ by taking $\alpha(g, x)$ to be the unique $h \in G$ such that $hf(x) = f(gx)$. Now we want to adjust this to become a homomorphism. Since G is Kazhdan, by Popa Superrigidity (6 A • 3), there exists a Borel map $b : X \to G$, a group homomorphism $\varphi : G \to G$, and a Borel subset $Z \subseteq Z$ with $\mu(Z) = 1$ such that—letting $f'(x) = b(x)f(x)$—for all $g \in G$, and $x \in Z$, $f'(gx) = \varphi(g)f'(x)$ (so we are unravelling the cocyle to get the homomorphism). Furthermore, we can suppose that Z is G-invariant (otherwise we take the intersection of all of the translates). Note that $G \curvearrowright \langle Z, \mu\rangle$ is (still) strongly mixing.

Claim 1

φ is a virtual embedding.

Proof .:.

Suppose $N = \ker\varphi$ is infinite. Since $G \curvearrowright \langle Z, \mu\rangle$ is strongly mixing, $N \curvearrowright \langle Z, \mu\rangle$ is ergodic. Since $f' : Z \to X$ is N-invariant, it follows that f' is μ-almost everywhere constant. But this means μ-almost every $x \in Z$ is sent by f to a single E-class, a contradiction with μ-nontriviality. ⊣

Thus $\varphi : G \to G$ is an automorphism. Let $\nu = f' * \mu$ be the probability measure on X defined by $\nu(A) = \mu(f'^{-1}"A)$.

Since $f'(gx) = \varphi(g)f'(x)$, it follows that ν is $\varphi(G)$-invariant. But $\varphi(G) = G$, so ν is G-invariant. Since $G \curvearrowright \langle X, \mu\rangle$ is uniquely ergodic, it follows that $\nu = \mu$. Thus $\mu(f'"(Z)) = \nu(f'"Z)$ which, by definition of the push-forward, is just $\mu(Z) = 1$. Since $f'"Z \subseteq G \cdot f"X$, we have that $\mu(G \cdot f(x)) = 1$. ⊣

§ 6 B. Weak Borel reductions

6 B • 1. Definition

Suppose that E, F are countable Borel equivalence relations on X, Y. Then E is *weakly Borel reducible* to F, written $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$, iff there exists a countable-to-one Borel homomorphism $f : X \to Y$ from E to F.

We remark the following about this definition.

1. If f is a Borel reduction, then f is a weak Borel reduction.
2. If $E \leqslant_{\mathrm{B}}^{\mathrm{w}} E'$ and $E' \leqslant_{\mathrm{B}}^{\mathrm{w}} E''$, then $E \leqslant_{\mathrm{B}}^{\mathrm{w}} E''$.
3. If $E \subseteq F$ are countable Borel equivalence relations on X, then id_X is a weak Borel reduction from E to F.

In essence, this is all that weakly Borel reductions are: Borel maps with inclusions.

6 B • 2. Theorem

If E, F are countable Borel equivalence relations on X, Y, then the following are equivalent:

(i) $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$;

(ii) There exists a countable Borel equivalence relation $R \supseteq E$ on X such that $R \leqslant_{\mathrm{B}} F$.

Proof .:.

For (ii) implies (i), we have that $E \leqslant_{\mathrm{B}}^{\mathrm{w}} R$ and $R \leqslant_{\mathrm{B}} F$. Hence $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$. For the other direction, suppose $f : X \to Y$ is a weak Borel reduction from E to F. Let $R = f^{-1}(F)$. Then $R \supseteq E$ is a countable Borel equivalence relation and f is a Borel reduction from R to F. ⊣

So we explore the connection between $\leqslant_{\mathrm{B}}^{\mathrm{w}}$ and $\leqslant_{\mathrm{B}}$.

6 B • 3. Definition

A countable Borel equivalence relation E is *weakly smooth* iff there is a smooth, countable, Borel F such that $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$.

We clearly have that all smooth Borel equivalence relations are weakly smooth, but we also have the reverse.

6 B • 4. Theorem

A countable Borel equivalence relation E is smooth iff it is weakly smooth.

Proof .:.

Let E be weakly smooth on X, and let F be a smooth countable Borel equivalence relation such that $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$. Then there exists a countable Borel $R \supseteq E$ on X such that $R \leqslant_{\mathrm{B}} F$. Thus R is smooth.

Hence there exists a Borel transversal T for R. By Feldman–Moore Theorem (2 • 2), there exists a countable group $G = \{g_n : n \in \omega\}$ and a Borel action$G \curvearrowright X$ such that $R = E_G^X$. Hence we can define a Borel selector $s : X \to X$ for E by $s(x) = g_n \cdot t$ where $T \cap [x]_R = \{t\}$ and n is minimal such that $g_n \cdot t \; E \; x$.

Thus E has a selector and so is also smooth. ⊣

6 B • 5. Definition

A countable Borel equivalence relation E is *weakly hyperfinite* iff there is a hyperfinite, countable, Borel F such that $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$.

Again, we have the same characterization as before.

6 B • 6. Theorem

If E is a weakly hyperfinite Borel equivalence relation on X, then E is hyperfinite.

Proof .:.

Let F be a hyperfinite, Borel equivalence relation such that $E \leqslant_{\mathrm{B}}^{\mathrm{w}} F$. Then there exists a countable Borel $R \supseteq E$ on X such that $R \leqslant_{\mathrm{B}} F$. Hence R is hyperfinite, and thus E is too. ⊣

So for id_X, there's no difference between $\leqslant_{\mathrm{B}}$ and $\leqslant_{\mathrm{B}}^{\mathrm{w}}$. Similarly, there's no difference for E_0.

6 B • 7. Theorem

A countable Borel equivalence relation E is *weakly universal* iff for every countable Borel equivalence relation F, $F \leqslant_{\mathrm{B}}^{\mathrm{w}} E$; equivalently $E_\infty \leqslant_{\mathrm{B}}^{\mathrm{w}} E$.

This yields to the following conjecture from 2001.

6 B • 8. Open Problem (Hjorth's Conjecture)

Every weakly universal countable Borel equivalence relation is universal.

After this, we arise at Thomas' question in the early 2000s: do there exist countable Borel equivalence relations $E \subseteq F$ such that $E \not\leqslant_{\mathrm{B}} F$? It turns out that there are.

6 B • 9. Theorem

There exist countable Borel equivalence relations E, F on a standard Borel X such that $E \subseteq F$ and $E \not\leqslant_{\mathrm{B}} F$.

Hence $\leqslant_{\mathrm{B}}$ and $\leqslant_{\mathrm{B}}^{\mathrm{w}}$ are in fact distinct notions. In proving this, as usual, the point is to find/Google suitable "big guns".

6 B • 10. Theorem

Suppose that G is a *proper* subgroup of finite index in $\mathrm{SL}_3(\mathbb{Z})$. Therefore,

1. G is a Kazhdan group.
2. G has no nontrivial, finite normal subgroups.
3. $\mathrm{SL}_3(\mathbb{Z})$ does not embed in G.

6 B • 11. Theorem

Suppose $H \curvearrowright \langle X, \mu \rangle$ is a strongly mixing, Borel action on a standard Borel probability space. Then there exists an H-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that the action of every infinite, finitely generated subgroup of H is uniquely ergodic.

Proof of Theorem 6 B • 9 .:.

Let $S = \mathrm{SL}_3(\mathbb{Z})$ and let $T = \ker\varphi$ where $\varphi : \mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{F}_7)$ acts surjectively and where $\mathbb{F}_7$ is the field with 7 elements (the '7' is unimportant, we just want a finite index). Then $1 < [S : T] < \aleph_0$. It follows that T is also finitely generated. Note that T is also a Kazhdan group.

Let $X \subseteq (2)^S$ be an S-invariant Borel subset with $\mu(X) = 1$ such that the action of every infinite, finitely generated subgroup of S on X is uniquely ergodic. Let $E \subsetneq F$ be the orbit equivalence relations of $T \curvearrowright X$ and $S \curvearrowright X$. We will show that $E \not\leq_{\mathrm{B}} F$.

Suppose that $f : X \to X$ is a Borel reduction from E to F. Then we can define a Borel cocycle $\alpha : T \times X \to S$ by taking $\alpha(t, x)$ to be the unique $s \in S$ such that $sf(x) = f(tx)$. By Popa Superrigidity (6 A • 3), since $T \subseteq S$ is Kazhdan, after deleting μ-null subset and adjusting f if necessary, we can suppose that $\alpha : T \to S$ is a group homomorphism.

Claim 1

$\alpha : T \to S$ is an embedding.

Proof .:.

Suppose not. Since $[\mathrm{SL}_3(\mathbb{Z}) : T] < \aleph_0$, T has non non-trivial, finite, normal subgroups. Thus $N = \ker\alpha$ is infinite. Since $S \curvearrowright \langle (2)^S, \mu\rangle$ is strongly mixing, $N \curvearrowright \langle X, \mu\rangle$ is ergodic. But then the N-invariant Borel map $f : X \to X$ is μ-almost everywhere constant, contradicting that we have a Borel reduction. ⊣

Also, since $T \ncong S$, it follows that $\alpha"T$ is a proper subgroup of S. Since the actions of S, T on $\langle X, \mu\rangle$ are free and $\alpha(t)f(x) = f(tx)$ for $t \in T$, $x \in X$, it follows that $f : X \to X$ is an injection. Thus we have an embedding of permutation groups: $\langle T, X\rangle \xrightarrow{\alpha, f} \langle S, X\rangle$. We more or less want f to be surjective, and to do this, we use unique ergodicity.

Hence we can define an $\alpha"T$-invariant probability measure $\nu = f * \mu$ on X by $\nu(A) = \mu(f^{-1}"A)$. Since $\alpha"T$ is finitely generated and infinite, $\alpha"T \curvearrowright \langle X, \mu\rangle$ is uniquely ergodic. Thus $\nu = \mu$ and hence $\mu(f"X) = \nu(f"X) = \mu(f^{-1}"(f"X)) = \mu(X) = 1$. So the map is onto.

As a proper subgroup of S, let $s \in S \setminus \alpha"T$. Then $\mu(f"X \cap s(f"X)) = 1$. Hence there exist $x, y \in X$ such that $f(x) = sf(y) \in f"X \cap s(f"X)$. Thus $f(x)\ F\ f(y)$ and so $x\ E\ y$. Hence there exists a $t \in T$ such that $x = ty$. It follows that $\alpha(t)f(y) = f(ty) = f(x) = sf(y)$. But then $s^{-1}\alpha(t)f(y) = f(y)$, which contradicts the fact that $S \curvearrowright X$ is free. ⊣

So that was the last consequence of Popa Superrigidity (6 A • 3) we will look at.

6 B • 12. Theorem (Miller)

If E is a countable, Borel equivalence relation on a standard Borel space X, then the following are equivalent.

(i) There is a universal countable Borel F on X such that $F \subseteq E$.

(ii) E is weakly universal.

Proof .:.

Showing (i) implying (ii) is easy, but (ii) implying (i) is very technical and never used. ⊣

6 B • 13. Corollary

$\equiv_{\mathrm{T}}$ is weakly universal.

Proof .:.

Let $\mathbb{F} = \langle a, b\rangle$ be the free group on two generators. Then E_∞ is the orbit equivalence relation of the shift action $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$. We can identify the left translation action $\mathbb{F}_2 \curvearrowright \mathbb{F}_2$ with the action $\Gamma \curvearrowright \mathbb{N}$ of a suitable group of recursive permutations. With this identification, $E_\infty \subseteq \equiv_{\mathrm{T}}$. ⊣

For later use, we record the following.

6 B • 14. Theorem

If E is a weakly universal, countable Borel equivalence relation, then E isn't essentially free.

Proof .:.

Otherwise, we can suppose that $E = E_H^Y$ for some free Borel action $H \curvearrowright Y$. But then there exists a countable G such that there's no μ-nontrivial Borel homomorphism from E_G to $E = E_H^Y$. And in this case, $E_G \not\leq_B^w E$. ⊣

6 B • 15. Open Problem

Is $\equiv_T$ countable universal?

For a more vague open problem, we have the following.

6 B • 16. Open Problem

Is there a "natural" action $\Gamma \curvearrowright 2^{\mathbb{N}}$ of a countable group such that $E_\Gamma^{2^{\mathbb{N}}}$ is $\equiv_T$?

We next try to develop an analog of ergodicity for $\equiv_T$.

Section 7. Martin's Measure

Recall the following definition.

7 • 1. Definition

For each $r \in 2^{\mathbb{N}}$, the corresponding *cone* is $C = \{s \in 2^{\mathbb{N}} : r \leqslant_T s\}$.

Note that if $\{C_n : n \in \omega\}$ is a countable set of cones, then $\bigcap_{n<\omega} C_n$ contains a cone. The following can be regarded as the analogue of ergodicity for $\equiv_T$. We have the following theorem due to Martin.

7 • 2. Theorem (Martin's Theorem)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X contains a cone, or $2^{\mathbb{N}} \setminus X$ contains a cone.

As a remark, X is $\equiv_T$-invariant iff whenever $y \equiv_T x \in X$, then $y \in X$. Before we prove this (from Borel determinacy), we state a few corollaries.

7 • 3. Corollary

If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel map, then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\varphi \restriction C$ is a constant map.

Proof .:.

For each $n \in \omega$, there exists an $\varepsilon_n \in \{0, 1\}$ such that $X_n = \{x \in 2^{\mathbb{Z}} : \varphi(x)_n = \varepsilon_n\}$ contains a cone C_n. Then $C = \bigcap_{n\in\omega} C_n$ contains a cone and $\varphi \restriction C$ is a constant function. ⊣

7 • 4. Corollary

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then the following are equivalent.

1. X contains a cone.
2. X is a $\leqslant_T$-cofinal.

Proof .:.

(1) implies (2) obviously. For (2) implies (1), clearly $2^{\mathbb{N}} \setminus X$ cannot contain a cone. ⊣

Proof of Martin's Theorem (7 • 2) .:.

Suppose X is a $\equiv_T$-invariant Borel subset of $2^{\mathbb{N}}$. Consider the 2 player game G_X where I wins iff $\langle s_n : n \in \omega \rangle \in X$ (here I plays $s_{2n} \in \{0,1\}$, and II plays $s_{2n+} \in \{0,1\}$ for $n < \omega$). Suppose, for example, that I has a winning strategy $\sigma : 2^{<\mathbb{N}} \to 2$. onsider the cone $C = \{t \in 2^{\mathbb{N}} : \sigma \leqslant_T t\}$. We claim that $C \subseteq X$.

Let $t = \langle t_n : n < \omega \rangle \in C$. Consider the play of G_X, where

- II plays $s_1, s_3, s_5, \cdots$, where $t = \langle s_{2n+1} : n < \omega \rangle$;
- I uses σ and plays $s_0, s_2, \cdots$.

Then $s = \langle s_n : n \in \omega \rangle \in X$ as σ is a winning strategy. Clearly $t \leqslant_T s$. Also, since $\sigma \leqslant_T t$, $s \leqslant_T t$. Thus $t \leqslant_T s \in X$, and so $t \in X$. ⊣

We will explore some consequences of Martin's Conjecture (MC).

7 • 5. Open Problem (Martin's Conjecture)

If $f : 2^{\mathbb{N}} \to 2^{\mathbb{Z}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following holds:

1. There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C into a single $\equiv_T$-class.
2. There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leqslant_T f(x)$ for all $x \in C$.

There is a stronger version (namely, the actual version), where (2) is replaced by

2'. There exists a cone $C \subseteq 2^{\mathbb{N}}$ and a countable $\alpha < \omega_1$ such that $f(x) \equiv_T x^{(\alpha)}$ for all $x \in C$, where $x^{(\alpha)}$ is the αth Turing jumnp.

Since the 1980s, there's been a *single* instance of progress on MC. Namely, the following theorem, proven some time in the 80s.

7 • 6. Theorem (Slaman–Steel)

Suppose that $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If there exists a $C \subseteq 2^{\mathbb{N}}$ such that $f(x) <_T x$ for all $x \in C$, then there exists a cone $D \subseteq C$ such that f sends D to a single $\equiv_T$-class.

Combining Martin's Theorem (7 • 2) and Slaman–Steel (7 • 6), if $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a counterexample to MC, then there exists a cone C such that $f(x)$ and x aren't $\leqslant_T$-comparable for all $x \in C$.

7 • 7. Theorem

(MC) If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following holds:

i. There exists a cone C such that f maps C to a single $\equiv_T$-class.

ii. There exists a cone C such that $f \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$. Furthermore, if $D \subseteq 2^{\mathbb{N}}$ is any cone, then $[f"D]_{\equiv_T} := \bigcup_{d \in D} [f(d)]_{\equiv_T}$ (the *saturation* of f) contains a cone.

7 • 8. Corollary

(MC) $\equiv_T <_B \equiv_T \oplus \equiv_T <_B \equiv_T \oplus \equiv_T \oplus \equiv_T <_B \cdots$.

Proof of Theorem 7 • 7 .:.

Suppose that (i) fails. By MC, there exists a cone C such that $x \leqslant_B f(x)$ for all $x \in C$. Clearly $f \restriction C$ is countable to one (since there are only countably many predecessors to any turing degree). Hence $f \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$.

Now we let D be any cone, and let $D_0 = C \cap D$. Since $f \restriction C$ is countable to one, it follows that $f"D_0$ is Borel. Hence $[f"D_0]_{\equiv_T}$ is also Borel. Since $[f"D_0]_{\equiv_T}$ is a $\equiv_T$-invariant, $\leqslant_T$-confinal, Borel subset of $2^{\mathbb{N}}$, Martin's Theorem (7 • 2) implies that $[f"D_0]_{\equiv_T}$ contains a cone. ⊣

As a matter of notation, for $x, y \in 2^{\mathbb{N}}$, then $x \oplus y$ is the usual recursive join: x is placed on the evens while y is placed on the odds.

7•9. Corollary

(MC) If $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_{\mathrm{T}}$-invariant, Borel subset, then the following are equivalent.

1. $\equiv_{\mathrm{T}} \restriction A$ is weakly universal.
2. A contains a cone.

Proof .:.

For (ii) $\to$ (i), suppose that A contians the cone $C = \{r \in 2^{\mathbb{N}} : z \leqslant_{\mathrm{T}} r\}$. We want to show that this is weakly universal. We can define an injective, weak Borel reduction from $\equiv_{\mathrm{T}}$ to $\equiv_{\mathrm{T}} \restriction A$ by $x \mapsto x \oplus z$.

For the other direction, suppose that $\equiv_{\mathrm{T}} \restriction A$ is weakly universal. Let $f : 2^{\mathbb{N}} \to A$ be a weak Borel reduction from $\equiv_{\mathrm{T}}$ to $\equiv_{\mathrm{T}} \restriction A$. By Theorem 7 • 7, $[f"2^{\mathbb{N}}]_{\equiv_{\mathrm{T}}} \subseteq A$ contains a cone. $\dashv$

Remark: there are currently no naturally occurring Borel sets of Turing degrees D for which it is known that $\equiv_{\mathrm{T}} \restriction D$ isn't weakly universal. In particular, it is not known when D is the set of minimal degrees.

7•10. Definition

Let E be a countable Borel equivalence relation on a standard Borel space X. Then $\equiv_{\mathrm{T}}$ is *E-m-ergodic* iff for every Borel homomorphism $f : 2^{\mathbb{N}} \to X$ from $\equiv_{\mathrm{T}}$ to E, there exists a cone C such that f maps C to a single E-class.

So by Martin's Theorem (7 • 2), $\equiv_{\mathrm{T}}$ is $\mathrm{id}_{2^{\mathbb{N}}}$-m-ergodic. One can also see that if $E \leqslant_{\mathrm{B}} F$ and $\equiv_{\mathrm{T}}$ is F-m-ergodic, then $\equiv_{\mathrm{T}}$ is E-m-ergodic.

7•11. Open Problem

Is $\equiv_{\mathrm{T}}$ E_0-m-ergodic?

But assuming MC, we understand everything.

7•12. Theorem

(MC) If E is a countable Borel equivalence relation, then exactly one of the following holds:

a. E is weakly universal.

b. $\equiv_{\mathrm{T}}$ is E-m-ergodic.

Proof .:.

If E is weakly universal, then there exists a weak reduction from $\equiv_{\mathrm{T}}$ to E; and so $\equiv_{\mathrm{T}}$ isn't E-m-ergodic. So (a) and (b) are mutually exclusive.

So it suffices to show that if $\equiv_{\mathrm{T}}$ isn't E-m-ergodic, then E is weakly universal. So suppose the Borel map $f : 2^{\mathbb{N}} \to X$ witnesses the failure of (b): a nontrivial, Borel homomorphism from $\equiv_{\mathrm{T}}$ to E. Since $\equiv_{\mathrm{T}}$ is weakly universal, there exists a weak Borel reduction $g : X \to 2^{\mathbb{N}}$ from E to $\equiv_{\mathrm{T}}$. Let $h = g \circ f$. Then h is a Borel homomorphism from $\equiv_{\mathrm{T}}$ to $\equiv_{\mathrm{T}}$.

Suppose there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that h sends C to a single $\equiv_{\mathrm{T}}$ class, say $[x]_{\equiv_{\mathrm{T}}}$. Then f sends a cone to the countable preimage of a single class: f maps C into the countable set $Y = g^{-1}"[x]_{\equiv_{\mathrm{T}}}$. We'd like to have the inverse image of one of the points is large, but this follows from the fact that there must be some $y \in Y$ with $f^{-1}(y)$ as $\leqslant_{\mathrm{T}}$-cofinal. By Martin's Theorem (7 • 2), it follows that $[f^{-1}(y)]_{\equiv_{\mathrm{T}}}$ contains a cone D. But then f maps D into $[y]_E$, a contradiction.

It follows that there exists a cone C such that $h \restriction C$ is countable to one. Thus $f \restriction C$ is countable to one, and $f \restriction C$ is a weak Borel reduction from $\equiv_{\mathrm{T}} \restriction C$ to E. Since $\equiv_{\mathrm{T}} \restriction C$ is weakly universal, E is weakly universal. $\dashv$

Finally, we give two striking applications of MC, which don't mention $\equiv_{\mathrm{T}}$. Recall Definition 4 • 23, that a Borel equivalence relation$E$ on X is Borel-Bounded iff every Borel $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ has a Borel homomorphism $\theta : X \to \mathbb{N}^{\mathbb{Z}}$ from E to $=^*$ (eventual equality) which dominates φ for all x: $\varphi(x) \leqslant^* \theta(x)$ for all $x \in X$.

Also recall Open Problem 4 • 25, where it's an open problem whether there exist any countable Borel equivalence relations that are not Borel-Bounded. MC answers the open problem by saying that any weakly universal one isn't Borel-Bounded.

7 • 13. Theorem

(MC) If E is weakly universal, then E is *not* Borel-Bounded.

Note that even assuming MC, *nothing* is known when $E_0 <_{\mathrm{B}} E$ isn't weakly universal. The theorem, however, only uses the fact that $\equiv_{\mathrm{T}}$ is E_0-m-ergodic under MC. So we're not even using the full power of MC. We will first prove the following special case.

7 • 14. Theorem

(MC) $\equiv_{\mathrm{T}}$ isn't Borel-Bounded.

7 • 15. Lemma

$=^*$ isn't weakly universal.

Proof .:.

It is easily seen that $=^*$ is Borel bireducible with E_0. Since E_0 is free, E_0 isn't weakly niversal. ⊣

Proof of Theorem 7 • 14 .:.

Identifying each $r \in 2^{\mathbb{N}}$ with the corresponding subset of $\mathbb{N}$, let $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel map such that

- if $r \cap 2\mathbb{N}$ is infinite, then $\varphi(r)$ is the strictly increasing enumeration of $r \cap 2\mathbb{N}$;
- otherwise $\varphi(r)$ is the identically zero function.

Now we claim the following.

Claim 1

For each $h \in \mathbb{N}^{\mathbb{N}}$, the $\equiv_{\mathrm{T}}$-invariant Borel set $S_h = \{r \in 2^{\mathbb{N}} : \exists s \in 2^{\mathbb{N}}\ (s \equiv_{\mathrm{T}} r \wedge h < \varphi(s))\}$ contains a cone.

Proof .:.

First fix a strictly increasing $e \in \mathbb{N}^{\mathbb{Z}}$ such that $h < e$. Now suppose $r \in 2^{\mathbb{Z}}$ satisfies $e \leqslant_{\mathrm{T}} r$. Consider $s \subseteq \mathbb{N}$ defined by

$$s = \{2e(n) : n \in \mathbb{N}\} \cup \{2\ell + 1 : \ell \in r\}.$$

Then clearly $s \equiv_{\mathrm{T}} r$, and $h < e < \varphi(s)$. ⊣

Finally, suppose that $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_{\mathrm{T}}$ to $=^*$ such that $\varphi(s) \leqslant^* \theta(s)$ for all $s \in 2^{\mathbb{N}}$. Since $=^*$ isn't weakly universal, $\equiv_{\mathrm{T}}$ is $-^*$-m-ergodic, and so there exists a cone C such that θ maps C to a single $=^*$-class, say, $[h]_{=^*}$. But this is a problem, because then $\varphi(s) \leqslant^* h$ for all $s \in C$ and so $C \cap S_h = \emptyset$, which is a contradiction. ⊣

To see that every weakly universal E isn't Borel-Bounded from MC, we we prove the following lemma.

7 • 16. Lemma

Suppose E, F are countable Borel equivalence relations, and that E is Borel-Bounded. Therefore,

(i) If $F \leqslant_{\mathrm{B}} E$, then F is Borel-Bounded.

(ii) If $F \subseteq E$, then F is Borel-Bounded.

Proof .:.

For (ii), suppose that $F \subseteq E$ are equivalence relations on X. Let $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ be any Borel map. Therefore there exists a Borel homomorphism $\theta : X \to \mathbb{N}^{\mathbb{N}}$ from E to $=^*$ such that $\varphi(x) \leqslant^* \theta(x)$ for all $x \in X$. So θ is also a Borel homomorphism from F to $=^*$.

For (i), suppose $f : X \to Y$ is a Borel reduction from F to E. Then $Z = \operatorname{im} f$ is a Borel subset of Y and there exists a Borel map $g : Z \to X$ such that $f \circ g = \mathrm{id} \restriction Z$.

Suppose that $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ is Borel. Ideally, we'l like to consider the function $g \circ \varphi$ (and 0 elsewhere on Y), but this won't work. Let $\Gamma = \{\gamma_n : n \in \omega\}$ be a countable group with a Borel action $\Gamma \curvearrowright X$ such that $E_\Gamma^X = F$ by Feldman–Moore Theorem (2 • 2). Let $\hat{\varphi} : Y \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by

$$\hat{\varphi}(z)(n) = \begin{cases} \max\{\varphi(\gamma_i g(z)) : i \leq n\} & \text{if } z \in Z \\ 0 & \text{otherwise.} \end{cases}$$

And now we just check that this works. Let $\hat{\theta} : Y \to \mathbb{N}^{\mathbb{N}}$ be a Borel homomorphism from E to $=^*$ such that $\hat{\varphi}(z) \leqslant^* \hat{\theta}(z)$ for all $z \in Y$ (since E is Borel-Bounded). Define $\theta : X \to \mathbb{N}^{\mathbb{Z}}$ by $\theta(x) = (\hat{\theta} \circ f)(x)$. Clearly θ is a Borel homomorphism from F to $=^*$. Now we just check domination.

Fix some $x \in X$ and let $z = f(x) \in Z$. Then there exists an $n \in \omega$ such that $x = \gamma_n g(z)$. So if m satisfies

- $m \geq n$; and
- $m \geq \max\{\ell : \hat{\varphi}(z)(\ell) > \hat{\theta}(z)(\ell)\}$;

then

$$\varphi(x)(m) = \varphi(\gamma_n g(z))(m) \leq \hat{\varphi}(z)(m) \leq \hat{\theta}(z)(m) = \hat{\theta}(f(x))(m) = \theta(x)(m).$$

Thus $\varphi(x) \leqslant^* \theta(x)$ for all $x \in X$. ⊣

So the following theorem tells us that measure is useless near the top of the countable Borel equivalence relations.

7 • 17. Theorem

(MC) Let E be a countable Borel equivalence relation on a standard Borel space X and let μ be a (not necessarily E-invariant) probability measure on X. Then there exists a Borel $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ isn't weakly universal.

We will make use of the following consequence of Borel–Cantelli (4 • 21).

7 • 18. Lemma

If $\langle X, \mu \rangle$ is a standard Borel probability space and $\theta : X \to \mathbb{N}^{\mathbb{N}}$ is Borel, then there exists an $h \in \mathbb{N}^{\mathbb{N}}$ such that $\mu(\{x \in X : \theta(x) \leqslant^* h\}) = 1$.

Proof .:.

Let $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map such that

- if $r \cap 2\mathbb{N}$ is infinite, then $\varphi(r)$ is the strictly increasing enumeration of $r \cap 2\mathbb{N}$; and
- otherwise, $\varphi(r)$ is identically 0.

By Feldman–Moore Theorem (2 • 2), there exists a countable group $\Gamma = \{\gamma_n : n \in \omega\}$ and a Borel action $\Gamma \curvearrowright 2^{\mathbb{N}}$ such that $E_\Gamma^{2^{\mathbb{N}}}$ is $\equiv_{\mathrm{T}}$. Let $\psi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by

$$\psi(x)(n) = \max\{\varphi(\gamma_n x) : m \leq n\}.$$

Then for all $r, s \in 2^{\mathbb{N}}$ with $s \equiv_{\mathrm{T}} r$, $\varphi(s) \leqslant^* \psi(r)$.

Let $f : X \to 2^{\mathbb{N}}$ be a weak Borel reduction from E to $\equiv_{\mathrm{T}}$. Define $\theta : X \to \mathbb{N}^{\mathbb{N}}$ by $\theta(x) = \psi(f(x))$. Then there exists an $h \in \mathbb{N}^{\mathbb{N}}$ such that $Y = \{x \in X : \theta(x) \leqslant^* h\}$ satisfies that $\mu(Y) = 1$ by Borel–Cantelli (4 • 21).

Let $Z = [f(x)]_{\equiv_{\mathrm{T}}}$. Therefore Z is Borel as f is countable-to-one, and saturation is Borel. Moreover, for each $r \in Z$, $\varphi(s) \leqslant^* h$ for all $s \equiv_{\mathrm{T}} r$. By an earlier claim, there exists a cone $C \subseteq 2^{\mathbb{N}} \setminus Z$. Hence Z doesn't contain a cone. By MC, $\equiv_{\mathrm{T}} \restriction Z$ isn't weakly universal. Since $E \leqslant_{\mathrm{B}}^{\mathrm{w}} \equiv_{\mathrm{T}} \restriction Z$, it follows that E isn't weakly universal. ⊣

Now return to consequences ZFC, where the remainder of the course takes place.

§ 7 A. Actual Theorems of ZFC

7 A • 1. Definition

f E, F are Borel equivalence relations on Polish spaces X, Y; write $E \leqslant_c F$ iff there exists a continuous (in the sense of X and Y) reduction from E to F.

A problem of Kanovei is to find *nontivial* instances of countable Borel equivalence relations E, F such that $E \leqslant_B F$ and $E \not\leqslant_c F$. What we mean by trivial is given by the following example.

7 A • 2. Example (Trivial Example)

Consider $\mathrm{id}_{[0,1]}$ and $\mathrm{id}_{2^{\mathbb{N}}}$. Then $\mathrm{id}_{[0,1]} \leqslant_B \mathrm{id}_{2^{\mathbb{N}}}$, but $\mathrm{id}_{[0,1]} \not\leqslant_c \mathrm{id}_{2^{\mathbb{N}}}$, just because of topological reasons.

Now in descriptive set theory, almost every equivalence relation is on a totally disconnected space.

7 A • 3. Theorem (Target Theorem 4)

$\equiv_T \not\leqslant_c E_\infty$.

7 A • 4. Definition

et $\equiv_1$ be the relation of recursive isomorphism on $2^{\mathbb{N}}$ (regarded as $\mathcal{P}(\mathbb{N})$; i.e. if $x, y \in 2^{\mathbb{Z}}$, then

$$x \equiv_1 y \quad \text{iff} \quad \text{there is a recursive permutation of } \mathbb{N} \text{ such that } \varphi''x = y.$$

7 A • 5. Theorem (Folklore Theorem)

The map $x \mapsto x'$ is a Borel reduction from $\equiv_T$ to $\equiv_1$.

Note that the following theorem implies that the above map is not continuous (as we will see later).

7 A • 6. Theorem (Target Theorem 5)

$\equiv_T \not\leqslant_c \equiv_1$.

It is an open problem whether $\equiv_1$ is universal. However, let $\mathrm{Rec}(\mathbb{N})$ be the group of recursive permutations of $\mathbb{N}$. Marks has shown that $E^{3^{\mathbb{N}}}_{\mathrm{Rec}(\mathbb{N})}$ is universal (the question is then why we can't with $E^{2^{\mathbb{N}}}_{\mathrm{Rec}(\mathbb{N})}$).

Both Target Theorem 4 (7 A • 3) and Target Theorem 5 (7 A • 6) are immediate consequences of the following.

7 A • 7. Theorem (Theorem AA)

Suppose G is a group of recursive permutations of $\mathbb{N}$ and $E = E^{2^{\mathbb{N}}}_G$. Then whenever $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to E, then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that θ maps C into a single E-class.

This is more generally true when $G \leqslant \mathrm{Sym}(\mathbb{N})$ is *any* countable subgroup. The proof just gets more technical, and less transparent.

7 A • 8. Theorem

If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, then the following are equivalent:

i. f is continuous.
ii. There exist $e \in \mathbb{N}$ and $z \in 2^{\mathbb{N}}$ such that $f(x) = \varphi_e^{z \oplus x}$ for all $x \in 2^{\mathbb{N}}$.

To introduce some notation, φ_e is the eth oracle Turing machine. If $r \in 2^{\mathbb{N}}$, then φ^r/e is the (partial) function computed by φ_e with oracle r.

Before proving it, we have the following corollary.

7 A • 9. Corollary

If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous, then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f(x) \leqslant_T x$ for all $x \in C$.

Proof of Theorem 7 A • 8 .:.

To show that (ii) implies (i), suppose that $f(x) = y$. We need to show that if you're close to x, then you're close to y. In particular, for $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $x_0 \restriction m = x \restriction m$, then (because we're only using finitely many values from the oracle)

$$f(x_0) \restriction n = \varphi_e^{z \oplus x_0} \restriction n = \varphi_e^{z \oplus x} \restriction n = f(x) \restriction n.$$

Then f is continuous.

Now suppose $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous. Let

$$z = \{\langle \tau, \sigma \rangle \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} : f^{-1}"U_\sigma \subseteq U_\tau\}$$

(here the U_σs re basic open sets). Since f is continuous, for all $x \in 2^{\mathbb{N}}$, if $f(x) = y$, then for all $n \in \mathbb{Z}$, there exists an $m \in \mathbb{N}$ such that $\langle x \restriction m, y \restriction n \rangle \in Z$. Thus y is computable from $z \oplus x$ as follows.

Given $\ell \in \mathbb{N}$, we search through z, x until we find $\langle \tau, \sigma \rangle \in z$ such that $\tau \subseteq x$ and $|\sigma| > \ell$. Then $y(\ell) = \sigma(\ell)$.⊣

The following definition is deceptively disgusting, but it is the proper notion to prove Theorem AA (7 A • 7).

7 A • 10. Definition

If $T \subseteq 2^{<\mathbb{N}}$ is a tree, then $[T] \subseteq 2^{\mathbb{N}}$ is the set of infinite branches.
A tree $T \subseteq 2^{<\mathbb{N}}$ is *perfect* iff every $t \in T$ has incompatible extensions.
A tree $T \subseteq 2^{<\mathbb{N}}$ is *pointed* iff T is perfect and $T \leqslant_{\mathrm{T}} x$ for every $x \in [T]$.

For example, $T = 2^{<\mathbb{N}}$ is clearly pointed, and so is every other recursive tree. Why do we care about such trees? Well, observe the following

7 A • 11. Result

If $T \subseteq 2^{<\mathbb{N}}$ is pointed, then for every $T \leqslant_{\mathrm{T}} z \in 2^{\mathbb{N}}$, there exists $x \in [T]$ such that $x \equiv_{\mathrm{T}} z$.

Proof .:.

For each $T \leqslant_{\mathrm{T}} z \in 2^{\mathbb{N}}$, let $x_z \in [T]$ be the branch which goes "left" at the nth branching point iff $z = 0$. Then $T \leqslant_{\mathrm{T}} x_z$ and so $z \leqslant_{\mathrm{T}} x_z$ (just by checking the path you take with respect to T, which x_z also computes). Also, since $T \leqslant_{\mathrm{T}} z$, we have that $x_z \leqslant_{\mathrm{T}} z$ and thus $x_z \leqslant_{\mathrm{T}} z$. ⊣

Thus, if T is a pointed tree, then $[T]$ is a "natural example" of a Borel set which contains a complete section of a cone. Now due to Martin, we have the following which gives examples of pointed trees. Note that this relies on Borel determinacy.

7 A • 12. Theorem

If $A \subseteq 2^{\mathbb{N}}$ is a $\leqslant_{\mathrm{T}}$-cofinal Borel subset, then there exists a pointed tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq A$.

Proof .:.

Consider the game where I and II alternate. I plays $x(0)$ then II plays $y(0)$, and then I plays $x(1)$, and so on: each turn I plays $x(n)$ and II plays $y(n)$ where $x(n), y(n) \in \{0, 1\}$. Here, II wins iff $y \in A$ and $x \leqslant_{\mathrm{T}} y$. Using Borel determinacy, we have the following.

Claim 1

IIhas a winning strategy.

Proof .:.

If not, then by Borel determinacy, I has a winning strategy $\tau : 2^{<\mathbb{N}} \to 2$. But then II can play any $\tau \leqslant_{\mathrm{T}} y \in A$. And then $x = \tau * y \leqslant_{\mathrm{T}} y$ and so II wins this play, contradicting that τ is a winning strategy.⊣

Let $\sigma : 2^{<\mathbb{N}} \to 2$ be a winning strategy for II. Let $\hat{\sigma} \in 2^{\mathbb{N}}$ be such that $\hat{\sigma} \equiv_{\mathrm{T}} \sigma$. For each $u \in 2^{<\mathbb{N}} \cup 2^{\mathbb{Z}}$ (as an abuse of notation) let $\sigma * u$ be the corresponding play of II using σ.

For $u \in 2^{<\mathbb{N}} \cup 2^{\mathbb{N}}$ and $x \in 2^{\mathbb{N}}$, say that the even part of u *agrees with* x if $u(2i) = x(i)$ for all $2i < |u|$.

We first define a perfect binary tree $T_1 \subseteq 2^{<\mathbb{N}}$ as follows. This tree will not be pointed, but when we look at the responses of II to plays in T_1, this will be the tree we're after.

- Let $u_\emptyset = \emptyset$.
- Next, let $u_{(0)}, u_{(1)}$ be the lexicographically least binary sequences whose even parts agree with $\hat{\sigma}$ such that $\sigma * u_{(0)}$ and $\sigma * u_{(1)}$ are incompatible.

To see that $u_{(0)}$ and $u_{(1)}$ exist, note that if $x \in 2^{\mathbb{Z}}$ then $x \leqslant_T \sigma * x$ and so the map $x \mapsto \sigma * x$ isn't constant on any $\leqslant_T$-unbounded set of reals, and so it must eventually split into some $u_{(0)}$ and $u_{(1)}$.

- Next, let $u_{(00)}$ and $u_{(01)}$ be the lexicographically least extensions of $u_{(0)}$ whose even parts agree with $\hat{\sigma}$ such that $\sigma * u_{(00)}, \sigma * u_{(01)}$ are incompatible.
- And continue in this fashion to create a binary tree.

Define

$$T_1 = \{s \in 2^{<\mathbb{N}} : \exists t \in 2^{<\mathbb{N}} (s \subseteq u_t)\}, \text{ and}$$
$$T = \{\sigma * u : u \in T_1\}.$$

Clearly T is a perfect binary tree by construction. Moreover, every branch of T is in A, since σ is a winning strategy for II: $[T] \subseteq A$. Also, $T \leqslant_T \sigma$ as σ was all that was used in the construction. Suppose that $y \in [T]$. We have to show that $T \leqslant_T y$. One can see that $y = \sigma * x$ for some $x \in [T_1]$. So the even part of y agrees with $\hat{\sigma}$ and hence $\sigma \leqslant_T x$. But since σ is winning for II, we have that $x \leqslant_T \sigma * x = y$, and so $T \leqslant_T \sigma \leqslant_T x \leqslant_T \sigma * x \leqslant_T y$, meaning T is pointed. ⊣

When we apply this theorem, we have something unbounded, and a pointed tree inside it But sometimes we want an "intelligent" pointed tree, which can do something for us in a proof. So now we show that we can do this.

7 A • 13. Result

If $T \subseteq 2^{<\mathbb{N}}$ is a pointed tree, and $T \leqslant_T z \in 2^{\mathbb{N}}$, then there exists a pointed subtree $T_0 \subseteq T$ such that $T_0 \equiv_T z$.

Proof .:.

Let T_0 be the subtree such that at each $2n$th branching point, we always go left if $z(n) = 0$, and always go right if $z(n) = 1$. Then clearly $T_0 \leqslant_T z$. Let $y \in [T_0]$ be the leftmost branch. Then $y \leqslant_T T_0$; and since $y \in [T]$, it follows that $T \leqslant_T y$ and therefore $T \leqslant_T T_0$. And it follows that $z \leqslant_T T_0$ just by looking at what happens at the even levels compared to T. Therefore $T_0 \equiv_T z$. Finally, suppose that $x \in [T_0] \subseteq [T]$. Therefore $T \leqslant_T x$, and so by the same idea (considering the even branching points compared to T), $z \leqslant_T x$ and therefore $T_0 \leqslant_T x$. ⊣

So just by refining the pointed tree given by Theorem 7 A • 12, we can get as complicated a pointed tree as we'd like.

7 A • 14. Definition

If $E \subseteq F$ are Borel equivalence relations on X, then F is *smooth over* E iff there exists a Borel homomorphism $\theta : X \to X$ from F to E such that $\theta(x)\ F\ x$ for all $x \in X$. (This implies θ is a Borel reduction from F to E).

In the proof of Theorem AA (7 A • 7), we will use the following. (For example, $\equiv_1 \subseteq \equiv_T$.)

7 A • 15. Theorem (Theorem BB)

If $H \leqslant \mathrm{Sym}(\mathbb{N})$ is any countable subgroup and $D \subseteq 2^{\mathbb{N}}$ is a cone such that $E_H^{2^{\mathbb{N}}} \upharpoonright D \subseteq \equiv_T \upharpoonright D$. Therefore $\equiv_T \upharpoonright D$ isn't smooth over $E_H^{2^{\mathbb{N}}} \upharpoonright D$.

Proving this is where pointed trees will come into play. But assuming this theorem, we can prove Theorem AA (7 A • 7).

Proof of Theorem AA (7 A • 7) .:.

Suppose $G \leqslant \mathrm{Sym}(\mathbb{N})$ is a group of recursive permutations and $\theta : 2^{\mathbb{Z}} \to 2^{\mathbb{N}}$ is a *continuous* homomorphism from $\equiv_T$ to $E_G^{2^{\mathbb{N}}}$. Since $E_G^{2^{\mathbb{N}}} \subseteq \equiv_T$, we can also regard θ as a homomorphism from $\equiv_T$ to $\equiv_T$.

Since θ is continuous, θ is computable on a cone. Explicitly, there exists a cone $C \subseteq 2^{\mathbb{Z}}$ such that $\theta(x) \leqslant_{\mathrm{T}} x$ for all $x \in 2^{\mathbb{N}}$. Applying Martin's Theorem (7 • 2), there exists a cone $D \subseteq C$ such that either

(i) $\theta(x) \equiv_{\mathrm{T}} x$ for all $x \in D$; or

(ii) $\theta(x) <_{\mathrm{T}} x$ for all $x \in D$.

By Theorem BB (7 A • 15), (i) cannot occur and thus (ii) holds. But then Slaman–Steel (7 • 6), there exists a cone $D' \subseteq D$ such that θ maps D' into a fixed $\equiv_{\mathrm{T}}$-class; say, $[z]_{\equiv_{\mathrm{T}}}$. Hence there exists a $y \in [z]_{\equiv_{\mathrm{T}}}$ such that $\theta^{-1}(y)$ is $\leqslant_{\mathrm{T}}$-cofinal. Now we can apply Martin's Theorem (7 • 2) to see that there exists a cone $D'' \subseteq [\theta^{-1}(y)]_{\equiv_{\mathrm{T}}}$. It follows that θ maps D'' into $[y]_{E_G^{2^{\mathbb{N}}}}$, which is what we wanted. ⊣

There are two major pieces of content: this nonsmoothness result of Theorem BB (7 A • 15) and of course Slaman–Steel (7 • 6).

Proof of Theorem BB (7 A • 15) .:.

Let $H \leqslant \mathrm{Sym}(\mathbb{N})$ be a countable subgroup and $D \subseteq 2^{\mathbb{Z}}$ be a cone such that $E_H^{2^{\mathbb{N}}} \restriction D \subseteq \equiv_{\mathrm{T}} \restriction D$. Suppose $\theta : D \to D$ is a Borelhomomorphism from $\equiv_{\mathrm{T}} \restriction D$ to $E_H^{2^{\mathbb{N}}} \restriction D$ such that $\theta(x) \equiv_{\mathrm{T}} x$ for all $x \in D$.

Since θ is countable to one (since it's going inside it's own equivalence class) it follows that $\theta"D$ is a Borel subset of $2^{\mathbb{N}}$. Also it is clear that $\theta"D$ is $\leqslant_{\mathrm{T}}$-cofinal. Applying Theorem 7 A • 12, there exists a pointed tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \theta"D$. In particular, it follows that if $x, y \in [T]$ then

$$x \equiv_{\mathrm{T}} y \quad \text{iff} \quad x\ E_H^{2^{\mathbb{N}}}\ y.$$

Let $H = \{h_n : n \in \omega\}$ and let $s \in 2^{\mathbb{N}}$ code the sequence $\langle h_n : n \in \omega\rangle$. Then after replacing T by a suitable pointed subtree by Result 7 A • 13, we can suppose that $s \leqslant_{\mathrm{T}} T$. Now we can do a simple diagonalization argument.

Let $x \in [T]$ be the leftmost branch. Then clearly $x \equiv_{\mathrm{T}} T$ as $T \leqslant_{\mathrm{T}} x$ as a pointed tree, and $x \leqslant_{\mathrm{T}} T$ as just the leftmost branch. Now define an increasing sequence of nodes $y_n \in T$ as follows.

- $y_0 = \emptyset$.
- For y_n defined, y_n^+ is such that $y_n \subseteq {y_n}^+ \in T$ is the next branching node. Let $|y_n^+| = \ell_n$. If $h_n(\ell_n) \notin x$, then let $y_{n+1} = y_n^{+\frown}1$. Otherwise, let $y_{n+1} = y_n^{+\frown}0$.

Taking $y = \bigcup_{n<\omega} y_n \in [T]$. Then $T \leqslant_{\mathrm{T}} y \leqslant_{\mathrm{T}} T \oplus x \oplus s \equiv_{\mathrm{T}} T$. Thus $y \equiv_{\mathrm{T}} T$. But by construction, $y \notin Hx$, a contradiction. ⊣.

Now we give some open pproblems before looking at some Borel combinatorics. First, consider the following definition and question due to Marks.

7 A • 16. Definition

A countable Borel equivalence relation E is *measure universal* iff for every countable Borel equivalence relation F on a standard Borel space X and any Borel probability measure μ on X, there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $F \restriction Y \leqslant_{\mathrm{B}} E$.

7 A • 17. Open Problem

Does there exist a measure universal E which *isn't* universal?

There are two results that suggest this might be interesting. The first, also due to Marks, is really quite weird.

7 A • 18. Theorem

(i) Recursive isomorphism on $3^{\mathbb{N}}$ is countable universal.

(ii) Recursive isomorphism on $2^{\mathbb{N}}$ is measure universal.

Using MC, Thomas has shown the following.

7 A • 19. Theorem

(MC) If F is any countable Borel equivalence relation on X and μ is any Borel probability measure, there exists a $Y \subseteq X$ with $\mu(Y) = 1$ such that $F \restriction Y$ isn't weakly universal.

Eventually, we will prove Theorem 7 A • 18, but it will require some Borel combinatorics.

Another open problem, practically an open area, is due to Marks.

7 A • 20. Observation

There are very few countable Borel equivalence relations E for which it is known that $E \leqslant_{\mathrm{B}} \equiv_{\mathrm{T}}$.

7 A • 21. Definition

Let $E_{T\infty}$ be the orbit equivalence relation for $\mathbb{F}_2 \curvearrowright (2)^{\mathbb{F}_2}$.

The following conjecture due to Marks is open.

7 A • 22. Open Problem

$E_{T\infty} \not\leqslant_{\mathrm{B}} \equiv_{\mathrm{T}}$.

A slightly stronger conjecture of Thomas is the following.

7 A • 23. Open Problem

If E is a nonhyperfinite countable Borel equivalence relation, then there exists a weakly universal countable Borel F such that $E \not\leqslant_{\mathrm{B}} F$.

Section 8. *Borel Combinatorics*

8 • 1. Definition

A *Borel graph* $\langle X, R\rangle$ consists of a standard Borel space X and a symmetric, irreflexive, Borel relation $R \subseteq X \times X$.

The first thing we will look at with these is chromatic number.

8 • 2. Definition

Let $\mathcal{G} = \langle X, R\rangle$ be a Borel graph. Then the *Borel chromatic number* $\chi_{\mathrm{B}}(\mathcal{G})$ is the least cardinality of a standard Borel space Y such that there exists a Borel map $c : X \to Y$ such that if $x \ R \ y$ then $c(x) \neq c(y)$. Such a map c is called a *Borel coloring*.

We need the Y to be contained in X so that we have a notion of Borel. Note that clearly $\chi(\mathcal{G})$, the actual chromatic number, is no more than $\chi_{\mathrm{B}}(\mathcal{G})$.

8 • 3. Example

Let $X = (2)^{\mathbb{Z}}$ and let $\mathcal{G}_0 = \langle X, E\rangle$ where $x \ E \ y$ iff $T(x) = y$ or $T^{-1}(x) = y$ where T generates the action $\mathbb{Z} \curvearrowright X$.

We can actually calculate the Borel chromatic number of $\mathcal{G}_0$.

8 • 4. Theorem

For $\mathcal{G}_0$ as in Example 8 • 3, $2 = \chi(\mathcal{G}_0) < \chi_{\mathrm{B}}(\mathcal{G}_0) = 3$.

Proof .:.

Since every connected component of $\mathcal{G}_0$ is a copy of $\mathbb{Z}$, it's clear that $\chi(\mathcal{G}_0) = 2$.

To see that $\chi_{\mathrm{B}}(\mathcal{G}_0) > 2$, suppose that $c : X \to 2$ is a Borel 2-coloring. For $i = 0, 1$, let $X_i = \{x \in X : c(x) = i\}$. Therefore the restriction of c to each connected component gives alternating colors on $\mathbb{Z}$. Thus $X = X_0 \sqcup X_1$ is a partition of X into two Borel subsets, each of which is invariant under $2\mathbb{Z} \curvearrowright X$. Let μ be the usual uniform product probability measure on X. Since $\mathbb{Z} \curvearrowright \langle X, \mu\rangle$ is strongly mixing, $2\mathbb{Z}$ acts ergodically on X. Therefore either X_0 or X_1 has measure 1. But then $\mu(X_0) = \mu(T(X_0)) = \mu(X_1)$ implies both have the same measure of 1, a contradiction.

The fact that $\chi_{\mathrm{B}}(\mathcal{G}_0) = 3$ follows from the next theorem. $\dashv$

As far as we know thus far, $\chi_{\mathrm{B}}(\mathcal{G}_0) \in [3, 2^{\aleph_0}]$. To prove that it is exactly 3, we need some notation for the next theorem.

8 • 5. Definition

If $\langle X, E\rangle$ is a graph and $x \in X$, then
- $E(x) = \{y \in X : y \mathrel{E} x\}$.
- $\deg(x) = |E(x)|$.

The next theorem is fairly easy if we leave out the "Borel".

8 • 6. Theorem

If $\mathcal{G} = \langle X, R\rangle$ is a Borel graph such that $\deg(x) \le k$ for all $x \in X$. Then $\chi_{\mathrm{B}}(\mathcal{G}) \le k + 1$.

First we need some basic results in Borel combinatorics.

8 • 7. Definition

A graph $\Gamma = \langle V, E\rangle$ is *locally finite* iff $\deg(v) < \infty$ for all $v \in V$.

8 • 8. Result

If $\mathcal{G} = \langle X, R\rangle$ is a locally finite Borel graph, then $\chi_{\mathrm{B}}(\mathcal{G}) \le \omega$.

Proof .:.

Let $\langle X, \mathcal{T}\rangle$ be a Polish space realizing the standard Borel structure of X. Let $\{\mathcal{U}_n : n \in \omega\}$ be a basis for the topology $\mathcal{T}$. Then we can define a Borel ω-coloring by taking $c(x)$ to be the least n such that $x \in U_n$ and $R(x) \cap U_n = \emptyset$. $\dashv$

This, of course, is not the most efficient way of proceeding, since the null graph $\langle X, \emptyset\rangle$ would use up countably many colors, for example.

8 • 9. Definition

Let $\Gamma = \langle V, E\rangle$ be a graph.
(i) A subset $D \subseteq V$ is *discrete* iff no two elements of D are joined by an edge.
(ii) A *maximal* discrete subset $D \subseteq V$ is called *kernel*

Of course, there is always a kernel, but the issue is whether there is a kernel that is *Borel*.

8 • 10. Result

If $\mathcal{G} = \langle X, R\rangle$ is a locally finite Borel graph, then there exists a Borel kernel $D \subseteq V$.

Proof .:.

For each Borel subset $Y \subseteq X$, let $R(Y) = \{x \in Y : \exists y \in Y\ (y \mathrel{R} x)\}$.

Claim 1

f $Y \subseteq X$ is Borel, then $R(Y)$ is Borel.

Proof .:.

Let $P = R \cap (X \times Y)$. Then P is a Borel subset of $X \times X$, and each section P_x countable (and actually finite as $\mathcal{G}$ is locally finite). By Theorem 2 • 6, $R(Y) = \mathrm{proj}_X(P)$ is Borel $\dashv$

Next, let $c : X \to \omega$ be a Borel ω-coloring; and for each $n \in \omega$, let $X_n = \{x \in X : c(x) = n\}$. Then $X = \bigsqcup_{n\in\omega} X_n$ is a partition of into discrete Borel subsets. We use this to inductively define our Borel kernel.

Define inductively Borel subsets $Y_n \subseteq X$ by

- $Y_0 = X_0$;
- $Y_{n+1} = Y_n \sqcup (X_{n+1} \setminus R(Y_n))$.

Then $Y = \bigcup_{n\in\omega} Y_n$ is a discrete Borel subset. To see that Y is a kernel, let $x \in X \setminus Y$. Then $x \in X_{n+1}$ for some $n \geq 0$. Since $x \notin Y_{n+1}$, it follows that $x \in R(Y_n)$. $\dashv$

And now we can prove Theorem 8 • 6.

Proof of Theorem 8 • 6 .:.

We argue by induction on $k \geq 0$. The result is clear when $k = 0$. Suppose the result holds for some $k > 0$. Let $\mathcal{G} = \langle X, R\rangle$ be a Borel graph such that $\deg(x) \leq k + 1$ for all $x \in X$. Let $Y \subseteq X$ be a Borel kernel, and let $Z = X \setminus Y$. Then $\deg_Z(v) \leq k$ for all $v \in Z$ (since everyone needs to be connected to someone in Y as otherwise Y wouldn't be maximal). Hence there exists a Borel $(k + 1)$-coloring $c_0 : Z \to \{0, \cdots, k\}$. Extend c_0 to a $(k + 2)$-coloring of X by $c(y) = k + 1$ for all $y \in Y$. $\dashv$

Let $\mathcal{G}_0 = \langle X, R\rangle$ be the graph associated with $\mathbb{Z} \curvearrowright (2)^{\mathbb{Z}}$ as before. Let $Y \subseteq X$ be a kernal. Then each connected component of $X \setminus Y$ is either an isolated point, or else a pair. So it is very easy to define a Borel 3-coloring of $\mathcal{G}_0$.

Next we start working towards the following theorem of Marks, which uses Borel determinacy.

8 • 11. Theorem

For each $n \geq 1$, there exists an n-regular acyclic Borel graph $\mathcal{G}$ with $\chi_B(\mathcal{G}) = n + 1$

8 • 12. Definition

A *marked group* is a group with a specified set $S_\Gamma \subseteq \Gamma \setminus \mathbf{1}$ of generators.

8 • 13. Definition

For Γ a marked group and X a standard Borel space, $G(\Gamma, X)$ is the Borel graph with vertex set

$$\mathrm{Free}(X^\Gamma) = \{y \in X^\Gamma : \gamma y \neq y \text{ for all } 1 \neq y \in \Gamma\}$$

and edge set E defined by $x\ E\ y$ iff $\exists\gamma \in S_\Gamma\ (\gamma x = y \vee \gamma y = x)$.

8 • 14. Definition

If Γ and Δ are marked groups, then $\Gamma * \Delta$ is the free product with generating set $S_\Gamma \cup S_\Delta$.

8 • 15. Theorem

If Γ, Δ are finitely generated marked groups, then

$$\chi_B(G(\Gamma * \Delta, \mathbb{N})) \geqslant \chi_B(G(\Gamma, \mathbb{N})) + \chi_B(G(\Delta, \mathbb{N})) - 1.$$

To try to understand this, consider $\Gamma = \Delta = C_2$. Then each connected component of $G(\Gamma, \mathbb{N})$ is just a pair. Thus $\chi_B(G(\Gamma, \mathbb{N})) = \chi(G(\Gamma, \mathbb{N})) = 2$. Each connected component of $G(\Gamma * \Delta, \mathbb{N})$ looks like $\mathbb{Z}$: it's just a line (but generated in a different way). By Theorem 8 • 15,

$$\chi_B(G(\Gamma * \Delta, \mathbb{N})) \geq 2 + 2 - 1 = 3.$$

Since each vertex has degree 2, $\chi_B(G(\Gamma * \Delta, \mathbb{N})) = 3$.

The only thing mysterious or "yucky" abou thte theorem is the appearance of $\mathbb{N}$. Fortunately, Seward–Tucker-Drob have shown that every finitely generated group Γ and all $n \geq 2$ have $\chi_B(G(\Gamma, n)) = \chi_B(G(\Gamma, \mathbb{N}))$. We, however, rely on $\mathbb{N}$ for the proof of Theorem 8 • 15.

8 • 16. Corollary

For each $n \geq 1$, let $\Gamma_n = C_2 * \cdots * C_2$ (n times). Therefore $G(\Gamma_n, \mathbb{N})$ is a cyclic n-regular graph with $\chi_{\mathrm{B}}(G(\Gamma_n, \mathbb{N})) = n + 1$.

Proof .:.

Clearly the Cayley graph of Γ_n is the n-regular tree. It is also clear that $\chi_{\mathrm{B}}(G(\Gamma_1, \mathbb{N})) = \chi_{\mathrm{B}}(G(C_2, \mathbb{N})) = 2$. Suppose inductively that $\chi_{\mathrm{B}}(G(\Gamma_n, \mathbb{N})) = n + 1$. Therefore by Theorem 8 • 15,

$$\begin{aligned}\chi_{\mathrm{B}}(G(\Gamma_{n+1}, \mathbb{N})) &= \mathrm{B}(G(\Gamma_n * C_2, \mathbb{N})) \\ &\geq \chi_{\mathrm{B}}(G(\Gamma_n, \mathbb{N})) + \chi_{\mathrm{B}}(G(C_2, \mathbb{N})) - 1 \\ &\geq (n+1) + 2 - 1 = n + 2.\end{aligned}$$

Since the graph of Γ_{n+1} is $(n+1)$-regular, it follows that $\chi_{\mathrm{B}}(G(\Gamma_{n+1}, \mathbb{N})) \leq n+2$. Hence $\chi_{\mathrm{B}}(G(\Gamma_{n+1}, \mathbb{N})) = n + 2$. ⊣

Theorem 8 • 15 is a consequence of the following main theorem of Marks.

8 • 17. Theorem (Marks' Main Theorem)

If Γ, Δ are nontrivial countable groups and $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ is Borel, then at least one of the following holds:

(i) There exists a continuous, injective Γ-equivariant ($\gamma x = y$ implies $\gamma f(x) = f(y)$) map $f : \mathrm{Free}(\mathbb{N}^{\Gamma}) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that $\operatorname{im} f \subseteq A$.

(ii) There exists a continuous injective Δ-equivariant ($\delta x = y$ implies $\delta f(x) = f(y)$) map $f : \mathrm{Free}(\mathbb{N}^{\Delta}) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that $\operatorname{im} f \subseteq \mathbb{N}^{\Gamma * \Delta} \setminus A$.

The proof of Marks' Main Theorem (8 • 17) uses Borel determinacy. In fact, Marks' Main Theorem (8 • 17) is *equivalent* to Borel determinacy modulo $\mathsf{Z}^- + \Sigma_1$-Replacement $+$ DC.

Before proving Marks' Main Theorem (8 • 17), we derive some consequences.

Proof of Theorem 8 • 15 .:.

Suppose that

$$\chi_{\mathrm{B}}(G(\Gamma, \mathbb{N})) = n + 1$$
$$\chi_{\mathrm{B}}(G(\Delta, \mathbb{N})) = m + 1.$$

Suppose that $c : G(\Gamma * \Delta, \mathbb{N}) \to (n + m)$ is a Borel $(n + m)$-coloring. Let

$$A = \{x \in G(\Gamma * \Delta, \mathbb{N}) : 0 \leq c(x) \leq n - 1\}.$$

Case 1. Suppose there exists a continuous, injective, Γ-equivariant $f : G(\Gamma, \mathbb{N}) \to G(\Gamma * \Delta, \mathbb{N})$ such that $\operatorname{im} f \subseteq A$. Suppose that $x, y \in \mathrm{Free}(\mathbb{N}^{\Gamma}) = G(\Gamma, \mathbb{N})$ are adjacent. Then without loss of generality, there exists a $\gamma \in S_{\Gamma}$ such that $y = \gamma \cdot x$. Since f is Γ-equivariant, $f(y) = f(\gamma \cdot x) = \gamma \cdot f(x)$ and so $f(x)$, $f(y)$ are adjacent and so $c(f(x)) \neq c(f(y))$. But this means $c \circ f$ is a Borel n-coloring of $G(\Gamma, \mathbb{N})$, a contradiction with the fact that $\chi_{\mathrm{B}}(G(\Gamma, \mathbb{N})) = n + 1$.

Case 2. By Marks' Main Theorem (8 • 17), there then exists an injective, continuous, Δ-equivariant map $f : G(\Delta, \mathbb{N}) \to G(\Gamma * \Delta, \mathbb{N})$ such that

$$\operatorname{im} f \subseteq G(\Gamma * \Delta, \mathbb{N}) \setminus A = \{x \in G(\Gamma * \mathbb{Z}) : n \leq c(x) \leq (n + m) - 1\}.$$

It follows as before that $c \circ f$ is a Borel m-coloring of $G(\Delta, \mathbb{N})$, a contradiction. ⊣

8 • 18. Definition

A graph $\langle V, E\rangle$ is *bipartite* iff there exists a partition $V = A \sqcup B$ such that every edge $e \in E$ joins a vertex $v \in A$ to a vertex $w \in B$.

8 • 19. Definition

A *perfect matching* of a graph $\langle V, E\rangle$ is a collection $M \subseteq E$ such that every vertex $v \in V$ lines on a unique edge $e \in M$.

8 • 20. Theorem (König's Theorem)

An n-regular, bipartite graph has a perfect matching.

8 • 21. Observation

There exists a 2-regular, bipartite, Borel graph with no *Borel* perfect matching.

Proof .:.

Let $\langle V, E\rangle$ be $\mathrm{Free}(\mathbb{Z}, 2^{\mathbb{Z}})$. Then every connected component has a graph that looks like $\mathbb{Z}$. So each element's edge is determined by the first choice of our edge in M, yielding an ability to choose elements. Formally, let $M \subseteq E$ be a perfect matching and let $<$ be a Borel linear ordering of V. Then

$$X_0 = \{v \in V : v \text{ is the } < \text{-least element of } v \in e \in M\}$$

is $2\mathbb{Z}$-invariant and we reach a contradiction as before. ⊣

Question: What about Borel-bipartite (meaning the pieces of the partition are Borel) graphs? Answer: the answer still is that there may not be a perfect matching. Consider the following theorem due to Marks.

8 • 22. Theorem

For every $n \geq 1$, there exists an n-regular, acyclic, Borel-bipartite graph with no Borel perfect matching.

To prove this, we make use of the following theorem.

8 • 23. Theorem

Let Γ, Δ be countable groups and let E_Γ, E_Δ be the orbit equivalence relations for the Borel actions $\Gamma \curvearrowright \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ and $\Delta \curvearrowright \mathrm{free}(\mathbb{N}^{\Gamma * \Delta})$. Then E_Γ, E_Δ do *not* have disjoint, Borel complete sections.

Proof .:.

Suppose that S, T are disjoint, Borel complete sections for E_Γ, E_Δ. We will apply Marks' Main Theorem (8 • 17) with $A = T$. Thus

$$S \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A = \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus T.$$

Case 1. Suppose there exists a continuous, injective, Γ-equivariant map $f : \mathrm{Free}(\mathbb{N}^{\Gamma}) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ with $\mathrm{im}\, f \subseteq A = T$. Then $\mathrm{im}\, f \neq \emptyset$ is a Γ-invariant subset such that $S \cap \mathrm{im}\, f = \emptyset$. Thus S is *not* a complete section for E_Γ.

Case 2. Otherwise, there exists a continuous, injective, Δ-equivariant map $f : \mathrm{Free}(\mathbb{N}^{\Delta}) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ with $\mathrm{im}\, f \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus T = S$. This implies as before that T is not a complete section for E_Δ, a contradiction. ⊣

Proof of Theorem 8 • 22 .:.

Let Γ, Δ be cyclic of $n \geq 1$. Let $Y \subseteq \left[\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})\right]^n$ be the standard Borel space consisting of the E_Γ-classes and E_Δ-classes.

Let $\mathcal{G}$ be the intersection graph on Γ i.e. $s, t \in Y$ are adjacent iff $s \cap t \neq \emptyset$. Note that each edge joins an E_Γ-class to an E_Δ-class and so $\mathcal{G}$ is Borel bipartite. Also $\mathcal{G}$ is clearly n-regular, since the $\Gamma * \Delta$-action is free.

Since $\mathcal{G}$ is bipartite, it contains no odd cycles. Suppose $\mathcal{G}$ contains an even cycle; say

$$s_1, t_1, s_2, t_2, \cdots, s_\ell, t_\ell$$

Without loss of generality, we can suppose the s_i are E_Γ-classes.

Let $\{x\} = s_1 \cap t_1$. Then there exists $1_\Delta \neq \delta_1 \in \Delta$ such that $\{\delta_1 x\} = t_1 \cap s_2$. Similarly, there is a $1_\Delta \neq \gamma_1 \in \Gamma$

such that $\{\gamma_1\delta_1 x\} = s_2 \cap t_2$. Continuing in this fashion, there exist $1_\Delta \neq \delta_i \in \Delta$ and $1_\Gamma \neq \gamma_i \in \Delta$ for $1 \leq i \leq \ell - 1$ such that

$$\gamma_{\ell-1}\delta_{\ell-1}\cdots\gamma_1\delta_1 x \in s_\ell \cap t_\ell.$$

But then as a cycle, there is a $1_\Delta \neq \delta_\ell \in \Delta$ such that

$$\delta_\ell\gamma_{\ell-1}\delta_{\ell-1}\cdots\gamma_1\delta_1 x = t_\ell \cap s_1.$$

But this is in the same Γ-orbit as x, and thus there is a $\gamma \in \Gamma$ such that

$$\gamma\delta_\ell\gamma_{\ell-1}\delta_{\ell-1}\cdots\gamma_1\delta_1 x = x,$$

contradicting the fact that $\Gamma * \Delta$ acts freely.

Now suppose that M is a Borel perfect matching of $\mathcal{G}$, and let

$$A = \{x \in \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) : \text{there exists an } \{s, t\} \in M \text{ such that } s \cap t = \{x\}\}.$$

Then A, $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A$ are disjoint, Borel complete sections for E_Γ, E_Δ, contradiction. $\dashv$

The following condition allows you to have disjoint complete sections.

8 • 24. Definition

Suppose that E_0, E_1 are countable Borel equivalence relations on a standard Borel space X.

- $E_0 \vee E_1$ is the smallest equivalence relation which contains E_0, E_1.
- E_0, E_1 are *everywhere nonindependent* iff whenever C is an $(E_0 \vee E_1)$-class, then there exists a sequence of distinct elements $x_0, \cdots, x_n \in C$ with $n \geq 1$ and a sequence $i_0, i_1, \cdots, i_n \in \{0, 1\}$ with $i_j \neq i_{j+1}$ for $j < n$ and $i_n \neq i_0$ such that

$$x_0 \ E_{i_0} \ x_1 \ E_{i_1} \ x_2 \cdots x_n \ E_{i_n} \ x_0.$$

Note that $E \vee F$ is a countable Borel equivalent relation. To see this, by Feldman–Moore Theorem (2 • 2), there exist countable groups G_0, G_1, and Borel actions $G_i \curvearrowright X$ such that $E = E^X_{G_0}$ and $F = E^X_{G_1}$. Let $G = G_0 * G_1$, and let $G \curvearrowright X$ be the corresponding action. Then $E \vee F = E^X_G$.

8 • 25. Open Problem

Let $\mathcal{G}_{fg}$ be the space of finitely generated groups, and let R be the Borel relation defined by $\Gamma \ R \ \Delta$ iff there exist isomorphic (unlabelled) Caley graphs of Γ and Δ.

R isn't an equivalence relation. Is the transitive closure of R Borel?

The proof of Marks' Main Theorem (8 • 17) will make use of the following theorem.

8 • 26. Theorem

If E_0, E_1 are everywhere non-independent countable Borel equivalence relations on the stadard Borel space X, then there exists a partition $X = X_0 \sqcup X_1$ such that each X_i is a Borel complete section for E_i.

The above theorem makes use of yet another result.

8 • 27. Definition

Let E be a countable Borel equivalence relation on X.

- $[E]^{<\infty}$ is the Borel subset of the standard Borel space $[X]^{<\infty}$ consisting of the nonempty finite $S \subseteq X$ such that S is contained in a single E-class.
- $\mathcal{G}_E = \langle [E]^{<\omega}, R\rangle$ where $S \ R \ T$ iff $S \cap T \neq \emptyset$.

8 • 28. Theorem

$\chi_{\mathrm{B}}(\mathcal{G}_E) \leq \omega$.

Proof .:.

First let $\langle g_n : n \in \omega\rangle$ be a sequence of Borel permutations $g_n : X \to X$ with $g_n^2 = 1$ such that if $x, y \in X$, then

$$x \; E \; y \leftrightarrow x = y \text{ or there exists } n \in \omega \text{ such that } g_n(x) = y.$$

Also fix some Borel linear order $<$ of X. Given $S \in [E]^{<\infty}$, let $S = \{x_0, \cdots, x_n\}$ where $x_0 < \cdots < x_n$. Then $c(S)$, the color of the set, is the lexicographically least sequence $\{k_{i,j}\}_{i \neq j}$ such that for all $i < j \leq n$, $g_{k_{i,j}} \cdot x_i = x_j$.

We will show that if $S \neq T \in [E]^{<\infty}$ with $S \cap T \neq \emptyset$, then $c(S) \neq c(T)$. So suppose $S \neq T$ are a counter-example. Then clearly $|S| = |T|$. Let

$$S = \{x_0, \cdots, x_n\}$$
$$T = \{y_0, \cdots, y_n\}$$

be the $<$-enumerations. Let $i, j \leq n$ be such that $x_i = y_j$.

Case 1. Suppose that $i \neq j$. Then without loss of generality, $i < j$. Hence $i < j$ implies $x_i < x_j = g_{k_{i,j}}(x_i)$. But this says that $y_j < g_{k_{i,j}}(y_j) = y_i$, which requires $j < i$, a contradiction.

Case 2. So we must have that $i = j$. But then for each $\ell \neq i$, we have $x_\ell = g_{k_{i,\ell}}(x_i) = g_{k_{i,\ell}}(y_i) = y_\ell$. And so $S = T$, a contradiction. ⊣

Proof of Theorem 8 • 26 .:.

Let $A \subseteq [E_0 \vee E_1]^{<\omega}$ be the Borel subset of finite $S \in [E_0 \vee E_1]^{<\infty}$ such that there exists an ordering $S = \{x_0, \cdots, x_n\}$ and a sequence $\{i_0, \cdots, i_n\}$ with $i_j \neq i_{j+1}$ and $i_n \neq i_0$ such that

$$x_0 \; E_{i_0} \; x_1 \; E_{i_1} \; x_2 \cdots x_n \; E_{i_n} \; x_0. \tag{$\star$}$$

Then A contains a finite, nonempty subset of each $(E_0 \vee E_1)$-class.

Let $c : [E_0 \vee E_1]^{<\omega} \to \omega$ be a Borel ω-coloring. Let $B \subseteq A$ be the Borel subset $S \in A$ such that whenever $T \in A$ lies in the same $(E_0 \vee E_1)$-class, then $c(S) \leq c(T)$. Then the elements of B are pairwise disjoint; and B still contains a finite, nonempty subset of each $E_0 \vee E_1$-class. We fix a Borel way of ordering each $S \in B$ as $x_0, \cdots, x_n$ and assigning a sequence $i_0, \cdots, i_n$ such that $(\star)$ holds.

For each $\varepsilon = 0, 1$, let $A_{\varepsilon,0}$ consist of those $x \in X$ such that there exist $S = \{x_0, \cdots, x_n\} \in B$ and $j \leq n$ such that $x = x_j$ and $i_j = \varepsilon$.

For example: if we start with $x_0 \; E_0 \; x_1 \; E_1 \; x_2 \; E_0 \; x_3 \; E_1 \; x_0$ then $x_0, x_2 \in A_{0,0}$ and $x_1, x_3 \in A_{1,0}$.

Clearly $A_{0,0} \cap A_{1,0} = \emptyset$. Also, note that for each $\varepsilon = 0, 1$, and $x \in A_{\varepsilon,0}$, there exists a $y \in [x]_{E_\varepsilon}$ such that $y \in A_{1-\varepsilon,0}$. We now inductively construct disjoint Borel sets $A_{0,n}, A_{1,n}$ satisfying:

(a) $A_{\varepsilon,n} \subseteq A_{\varepsilon,n+1}$,
(b) for each $x \in A_{\varepsilon,n}$, there exists a $y \in [x]_{E_\varepsilon}$ such that $y \in A_{1-\varepsilon,n}$.

Case 1. Suppose n is even. Then we define

$$A_{0,n+1} = A_{0,n} \cup \left([A_{0,n}]_{E_0} \setminus A_{1,n}\right)$$
$$A_{1,n+1} = A_{1,n} \cup \left([A_{0,n}]_{E_1} \setminus A_{0,n+1}\right).$$

Clearly $A_{0,n+1}$, and $A_{1,n+1}$ are disjoint and (a) holds. To see that (b) holds, first suppose that $x \in A_{0,n+1} \setminus A_{0,n}$. Then there exists (by saturation by E_0) an $x' \in A_{0,n}$ such that $[x]_{E_0} = [x']_{E_0}$, and by induction there exists a $y \in A_{1,n} \subseteq A_{1,n+1}$ such that $y \in [x']_{E_0} = [x]_{E_0}$. Next, suppose that $x \in A_{1,n+1} \setminus A_{1,n}$. As it's in the saturation of the previous one, there exists a $y \in A_{0,n} \subseteq A_{0,n+1}$ such that $[x]_{E_1} = [y]_{E_1}$. Thus (b) holds.

Case 2. Suppose n is odd. Then we define

$$A_{0,n+1} = A_{0,n} \cup \left([A_{1,n}]_{E_0} \setminus A_{1,n}\right)$$
$$A_{1,n+} = A_{1,n} \cup \left([A_{1,n}]_{E_1} \setminus A_{0,n+1}\right).$$

Arguing as above, the inductive hypotheses still hold.

Let $X_0 = \bigcup_{n\in\omega} A_{0,n}$ and $X_1 = \bigcup_{n\in\omega} A_{1,n}$. Then it suffices to prove the following claims.

Claim 1

$X = X_0 \sqcup X_1$.

Claim 2

If $\varepsilon = 0, 1$, then X_ε is a complete Borel section for $E_{1-\varepsilon}$.

Proof .:.

Assuming Claim 1, consider the case where $\varepsilon = 0$. Let $x \in X$. If $x \in X_0$, then clearly $[x]_{E_1} \cap X_0 \neq \emptyset$. Otherwise, by Claim 1, $x \in X_1$ and hence there exists a $y \in [x]_{E_1}$ such that $y \in X_0$. ⊣

Proof of Claim 1 .:.

Suppose $z \in X$. Then there exists $x_0 \in A_{0,0} \cap [z]_{E_0 \vee E_1}$. It follows that there exists a sequence

$$x_0 \; E_{i_0} \; x_1 \; E_{i_1} \; x_2 \cdots x_m \; E_{i_m} \; z,$$

for $i_\ell \in \{0,1\}$. We can suppose inductively that $x_m \in X_0 \cup X_1$. Suppose, for example, that $x_m \in X_0$. Then there exists an even n such that $x_m \in A_{0,n}$. If $z \in A_{0,n} \cup A_{1,n}$, we are done. If not, and $z \in [A_{0,n}]_{E_0}$, then $z \in A_{0,n+1}$. Otherwise, $z \in [A_{0,n}]_{E_1}$ and so $z \in A_{1,n+1}$. ⊣

⊣

So we are almost ready to start proving Marks' Main Theorem (8 • 17). Before finally beginning the proof, we need one more technical lemma.

8 • 29. Definition

Let $Y \subseteq \mathbb{N}^{\Gamma * \Delta}$ be the set of all elements $y \in \mathbb{N}^{\Gamma*\Delta}$ such that for all $g \in \Gamma * \Delta$,

$$\gamma \cdot gy \neq gy \text{ for all } \gamma \in \Gamma \setminus \{1_\Gamma\}, \text{ and}$$
$$\delta \cdot gy \neq \delta y \text{ for all } \delta \in \Delta \setminus \{1_\Delta\}.$$

Note that Y is $\Gamma * \Delta$-invariant. Clearly, $\mathrm{Free}(\mathbb{N}^{\Gamma*\Delta}) \subseteq Y$. So what concerns us is the difference.

Let E_Γ be the Γ-orbit equivalence relation on $\mathbb{N}^{\Gamma*\Delta}$ and E_Δ be the Δ-orbit equivalence relation.

8 • 30. Lemma

$E_\Gamma \restriction (Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta}))$ and $E_\Delta \restriction (Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta}))$ are everywhere nonindependent.

Proof .:.

Let $C \subseteq Y \setminus \mathrm{free}(\mathbb{N}^{\Gamma*\Delta})$ be a $(\Gamma * \Delta)$-orbit. Then there exists an $x \in C$ and $g \in (\Gamma * \Delta) \setminus (\Gamma \cup \Delta)$ such that $g \cdot x = x$. Suppose, for example, that $g = \gamma_1 \delta_1 \cdots \gamma_n \delta_n \gamma$ where $\gamma_i \in \Gamma \setminus \{1_\Gamma\}$, $\delta_i \in \Delta \setminus \{1_\Delta\}$ for $1 \leq i \leq n$ and $\gamma \in \Gamma$. We also suppose n is minimal within C.

Case 1. Suppose $\gamma = 1$. Then $x, \delta_n x, \gamma_n \delta n x, \ldots, \delta_1 \cdots \gamma_n \delta_n x, \gamma_1 \delta_1 \cdots \delta_n x = x$ witnesses nonidependence.

Case 2. Suppose $\gamma \neq 1$. Suppose $\gamma\gamma_1 \neq 1$. Then replacing γ_1 by $\gamma_1' = \gamma\gamma_1$ and x by $x' = \gamma x$, we obtain $\gamma_1' \delta_1 \cdots \gamma_n \delta_n x' = x'$. And so we are in (Case 1).

Case 3. Suppose $\gamma\gamma_1 = 1$. Then replacing x by $x' = \gamma x$, we obtain $\delta_1 \gamma_2 \delta_2 \cdots \delta_{n-1} \gamma_n \delta_n \cdot x' = x'$. By minimality of n, we have a contradiction. ⊣

Now we will prove Marks' Main Theorem (8 • 17).

Proof of Marks' Main Theorem (8 • 17) .:.

Each non-identity element of $\Gamma * \Delta$ can be uniquely written as a finite product of the form

(i) $\gamma_{i_0}\delta_{i_1}\gamma_{i_2}\cdots$; or
(ii) $\delta_{i_0}\gamma_{i_1}\delta_{i_2}\cdots$;

where $\gamma_{i_j} \in \Gamma \setminus \{1\}$, and $\delta_{i_j} \in \Delta \setminus \{1\}$. Words of the form (1) are called Γ-*words* and words of the form (ii) are called Δ-*words*. We will make use of games for building an element $y \in \mathbb{N}^{\Gamma*\Delta}$, where I decides y on Γ-words, and II* decides y on Δ-words.

First we fix injective (possibly finite) listings $\gamma_0, \gamma_1, ...$; and $\delta_0, \delta_1, ...$ of $\Gamma \setminus \{1\}$ and $\Delta \setminus \{1\}$ respectively.

Next we define the turn function $t : (\Gamma * \Delta) \setminus \{1\} \to \mathbb{N}$ as follows. Suppose $\alpha \in (\Gamma * \Delta) \setminus 1$ has the form (i) or (ii) with associated sequence $i_0, i_1, ..., i_m$ (where the indices are from these fixed enumerations). Then $t(\alpha)$ is the least n such that $i_j + j \le n$ for all $j \le m$.

For example, the elements with $t(\alpha) = 0$ are γ_0, δ_0. The elements with $t(\alpha) = 1$ are $\gamma_1, \gamma_0\delta_0, \gamma_1\delta_0$, and similarly $\delta_1, \delta_0\gamma_0$, and $\delta_1\gamma_0$. And so on.

Write e for $1_{\mathbb{N}^{\Gamma*\Delta}}$. For each Borel subset $B \subseteq Y$ and $k \in \mathbb{N}$, the following game G_k^B produces an element $y \in \mathbb{N}^{\Gamma*\Delta}$ with $y(e) = k$. First we set $y(e) = k$. On the nth turn of G_k^{B}, first I defines $y(\alpha)$ on the Γ-words with $t(\alpha) = n$; and II defines $y(\alpha)$ on the Δ-words with $t(\alpha) = n$.

The winning conditions for G_k^B:

- If $y \in Y$, then II wins iff $y \in B$.

Suppose that $y \notin Y$. Then there exists an $\alpha \in \Gamma * \Delta$ such that either $\exists\gamma \in \Gamma \setminus \{1\}(\gamma\alpha^{-1}y = \alpha^{-1}y)$ or $\exists\delta \in \Delta \setminus \{1\}(\delta\alpha^{-1}y = \alpha^{-1}y)$. In the former case, we say that (α, Γ) witnesses that $y \notin Y$, and in the latter case, (α, Δ) witnesses that $y \notin Y$. In both cases, we say that α witnesses that $y \notin Y$.

- If $y \notin Y$ and (e, Γ) witnesses that $y \notin Y$ then I loses.
  Otherwise, if (e, Δ) witnesses that $y \notin Y$, then II loses.
  If neither of the above cases hold, then I wins iff there is a Δ-word α witnessing that $y \notin Y$ such that for all Γ-words β with $t(\beta) \le t(\alpha)$, β doesn't witness that $y \notin Y$.

Finally, recall that $E_\Gamma \restriction (Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta}))$ and $E_\Delta \restriction (Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta}))$ are everywhere nonindependent. Hence there exists a Borel subsets $C \subseteq Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta})$ such that C meets every E_Δ-class on $Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta})$ and the complement C^{c} meets every E_Γ-class.

Now let $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta})$ be Borel. Then we define $B_A = A \cup C \subseteq Y$. So by Borel Determinacy, for each $k \in \mathbb{N}$, either I or II has a winning strategy in the game $G_k^{B_A}$. Hence one of the players has a winning strategy for infinitely many $k \in \mathbb{N}$.

Suppose, for example, that $S = \{k \in \mathbb{N} : \text{II has a winning strategy in } G_k^{B_A}\}$ is infinite. Clearly there exists an injective, continuous, Γ-equivariant map from $\mathrm{Free}(\mathbb{N}^\Gamma)$ to $\mathrm{Free}(S^\Gamma)$. Hence it's enough to show that there is an injective, continuous, Γ-equivariant map $f : \mathrm{Free}(S^\Gamma) \to \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta})$ such that $\operatorname{im} f \subseteq A$.

We will define f such that for all $x \in \mathrm{Free}(S^\Gamma)$, the following hold:

(i) $f(x)(\gamma) = x(\gamma)$ for all $\gamma \in \Gamma$;
(ii) $f(x)$ will be a winning outcome for II's winning strategy in $G_{x(e)}^{B_A}$.

Clearly (i) ensures that f is injective. Suppose, for the moment, that f is Γ-equivariant and that $f(x) \in Y$ for all $x \in \mathrm{Free}(S^\Gamma)$. I will help to ensure that these are both true.

Claim 1

With the suppositions above, $f(x) \in A$ for all $x \in \mathrm{Free}(S^\Gamma)$.

Proof .:.

Since $f(x) \in Y$, and and $f(x)$ is an outcome in II's winning strategy in $G^{B_A}_{x(e)}$, it follows that $f(x) \in B_A = A \cup C$. Suppose that $f(x) \in C \subseteq Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$. Since $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ is Γ-invariant, it follows that if $\gamma \in \Gamma$, then $\gamma f(x) = f(\gamma x) Y$ and in fact is in C. And so $\Gamma \cdot f(x) \subseteq C$. But this contradicts the fact that every Γ-orbit on $Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ intersects C^c. ⊣

The basic idea: in order to deal with the assumptions for a single x, while II uses the winning strategy in the games $G^{B_A}_{x(\gamma^{-1})}$, I plays to ensure that f is Γ-equivariant and that $f(x)(\gamma) = x(\gamma)$.

Finally fix some $x \in \mathrm{Free}(S^\Gamma)$. Then for each $\gamma \in \Gamma$, we will play an instance of $G^{B_A}_{x(\gamma^{-1})}$ to produce an element $y \in \mathbb{N}^{\Gamma * \Delta}$ which will be equal to $\gamma f(x)$. We begin by setting $\gamma f(x)(e) = x(\gamma^{-1})$ for each $\gamma \in \Gamma$ (we're playing infinitely many games simultaneously).

Now suppose that $\gamma f(x)(\alpha)$ is defined for all $\gamma \in \Gamma$ and $\alpha \in \Gamma * \Delta$ with $t(\alpha) < n$. Then the nth move of I in each game is as follows. Suppose β is a Γ-word with $t(\beta) = n$. Then $\beta = \gamma_i \alpha$ where $i \leq n$ and $t(\alpha) < n$. For every $\gamma \in \Gamma$, we define $\gamma f(x)(\gamma_i \alpha)$ to be $(\gamma_i^{-1} \gamma f(x))(\alpha)$, which has been defined. Then II uses the winning strategy in all of these games to define $(\gamma f(x))(\delta_i \alpha)$ for every Δ-word $\delta_i \alpha$ with $t(\delta_i \alpha) = n$. Clearly f is Γ-equivariant by the way I has played, and $f(x)(\gamma) = x(\gamma)$, as a special case of Γ-equivariance. Also, it is clear that f is continuous, since if the words agree on large initial segments, then the players play along those long initial segments. Thus it only remains to check that $f(x) \in Y$ for all $x \in \mathrm{Free}(S^\Gamma)$.

First, since $x \in \mathrm{Free}(S^\Gamma)$ and $f(x) \restriction \Gamma = x \restriction \Gamma$, we have that $\gamma f(x) \neq f(x)$ for all $\gamma \in \Gamma \setminus \{1\}$. Thus (e, Γ) doesn't witness that $f(x) \notin Y$. Since $f(x)$ is an outcome for II's winning strategy, (e, Δ) doesn't witness $f(x) \notin Y$. Now we prove inductively that α doesn't witness $f(x) \notin Y$ for all $x \in \mathrm{Free}(S^\Gamma)$ and all $\alpha \in \Gamma * \Delta$ with $t(\alpha) = n$. This is certainly true for the identity, so now we proceed by inductionon n. First, suppose $\alpha = \gamma\beta$ is a Γ-word with $t(\alpha) = n$ and $t(\beta) < n$. Since

$$\alpha^{-1} f(x) = \beta^{-1} \gamma^{-1} f(x) = \beta^{-1} f(\gamma^{-1} x),$$

and β doesn't witness that $f(\gamma^{-1} x) \notin Y$, it follows that α doesn't witness $f(x) \notin Y$. Let α be a Δ-word with $t(\alpha) = n$. Hence we can suppose that if β is a Γ-word with $t(\beta) \leq n$, then β doesn't witness that $f(x) \notin Y$. Since $f(x)$ is an outcome of II's winning strategy, it cannot be that α is the first time that a witness appears (as that means that II would lose). It follows that α cannot witness that $f(x) \notin Y$. Hence $f(x) \in Y$, as desired. This completes the proof, as the case is symmetric where S is the $k \in \mathbb{N}$ where I has a winning strategy for $G^{B_A}_k$. ⊣

§ 8 A. Recursive Isomorphism

We now state another theorem of Marks (recall $\mathbb{F}_2$ is the free group on two generators).

8 A • 1. Theorem

Let $G \leqslant \mathrm{Sym}(\mathbb{F}_2 \times \mathbb{N})$ be a countable group of permutations such that for each $g \in \mathbb{F}_2$, there exists $\rho_g \in G$ such that $\rho_g(h, n) = \langle gh, n \rangle$ for all $\langle h, n \rangle \in \mathbb{F}_2 \times \mathbb{N}$. Then

(i) $G \curvearrowright 2^{\mathbb{F}_2 \times \mathbb{N}}$ is a measure universal countable Borel equivalence relation.

(ii) $G \curvearrowright 3^{\mathbb{F}_2 \times \mathbb{N}}$ is a universal Borel equivalence relation.

It's open whether the action in (1) is universal. As stated, of course, this is hard to really understand. So instead we have the following corollary.

8 A • 2. Corollary

(i) Recursive isomorphism on $2^{\mathbb{N}}$ is measure universal.

(ii) Recursive isomorphism on $3^{\mathbb{N}}$ is countable universal.

Proof .:.

Let $Y \in \{2,3\}$. Via computable bijection between $\mathbb{N}$ and $\mathbb{F}_2 \times \mathbb{N}$, we can identify $2^{\mathbb{N}}$ and $2^{\mathbb{F}_2 \times \mathbb{N}}$. Then $G = \operatorname{Rec}(\mathbb{F}_2 \times \mathbb{N})$ (recursive permutations of $\mathbb{F}_2 \times \mathbb{N}$) satisfies the hypotheses of Theorem 8 A • 1. ⊣

Proof of Theorem 8 A • 1 .:.

Throughout, $Y \in \{2,3\}$, as much will be the same. Let E_∞ be the universal countable Borel equivalence relation arising from $\mathbb{F}_2 \curvearrowright X$ where $X = 2^{\mathbb{F}_2}$. Let $f : X \to Y^{\mathbb{N}}$ be *any* Borel map. We modify this to get a map from X to $Y^{\mathbb{F}_2 \times \mathbb{N}}$.

Consider the associated $\hat{f} : X \to Y^{\mathbb{F}_2 \times \mathbb{N}}$ defined by $\hat{f}(x)(h,n) = f(h^{-1}x)(n)$. Suppose $x, y \in X$ and $g \in \mathbb{F}_2$ satisfies $gx = y$. Then

$$(\rho_g \cdot \hat{f}(x))(h,n) = \hat{f}(x)(g^{-1}h,n) = f(h^{-1}gx)(n) = \hat{f}(gx)(h,n) = \hat{f}(y)(h,n).$$

Thus $\rho_g \hat{f}(x) = \hat{f}(gx) = \hat{f}(y)$, and $\hat{f}$ is a Borel homomorphism from $\mathbb{F}_2 \curvearrowright X$ to $\mathbb{F}_2 \curvearrowright Y^{\mathbb{F}_2 \times \mathbb{N}}$.

We will define an injection $f : X \to Y^{\mathbb{N}}$ so that $\hat{f}$ witnesses either (i) or (ii) of Theorem 8 A • 1.

Basic Idea: we will construct f such that for all $\rho \in G$ and $x \in X$, either

- $\rho \cdot \hat{f}(x) \notin \operatorname{im} \hat{f}$; or
- $\rho \cdot \hat{f}(x) = \hat{f}(y)$ for some $y \; E_\infty \; x$.

We now introduce a definition to help with this.

8 A • 3. Definition

- $\rho \in G$ has type I iff for every $k > \omega$, there exist $m, n > k$ with $n \neq m$ such that $\rho^{-1}(1,n) \in \mathbb{F}_2 \times \{m\}$.
- $\rho \in G$ has type II if it is not type I, and there exists an m such that for infinitely many $n \in \mathbb{N}$, $\rho^{-1}(1,n) \in \mathbb{F}_2 \times \{m\}$.
- $\rho \in G$ has type III iff it is not type I nor type II.

Note that if $g \in \mathbb{F}_2$, then ρ_g has type III. If $\rho \in G$ has type III, then for all but finitely many $n \in \mathbb{N}$, $\rho^{-1}(1,n) \in \mathbb{F}_2 \times \{n\}$.

For each $\rho \in G$ of type II, fix some m_ρ such that there exist infinitely many $n \in \mathbb{N}$ with $\rho^{-1}(1,n) \in \mathbb{F}_2 \times \{m_\rho\}$. We say that each such n *witnesses* that ρ has type II.

To begin, we inductively construct a partition $\mathbb{N} = S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3$ such that S_2 and S_3 are infinite, and for every $\rho \in G$,

- if ρ has type I, then there exist an $n \in S_1$ and $m \in S_0$ such that $\rho^{-1}(1,n) \in \mathbb{F}_2 \times \{m\}$;
- if ρ has type II, then there are infinitely many $n \in S_2$ and infinitely many $n \in S_3$ such that n witnesses that ρ has type II.

Now we make another definition, and then we actually prove something.

8 A • 4. Definition

A set $S \subseteq \mathbb{N}$ is *good* if whenever $\rho \in G$ has type II, then infinitely many $n \in S$ witness this.

Hence S_2 and S_3 are good by definition. Furthermore, any good subset can be partitioned into two good subsets. During the proof, we will successively add more constraints to the map $f : X \to Y^{\mathbb{N}}$.

Constraint 1. For every $x \in X$, define $f(x)(m) = \begin{cases} 0 & \text{if } m \in S_0 \\ 1 & \text{if } n \in S_1. \end{cases}$

Claim 1

If ρ has type I, then $\rho \hat{f}(x) \notin \operatorname{im} \hat{f}$ for all $x \in X$.

Proof .:.

There exists an $n \in S_1$ and $m \in S_0$ such that $\rho^{-1}(1,n) = \langle h, m\rangle$ for some $h \in \mathbb{F}_2$. Hence
$$(\rho \cdot \hat{f}(x))(1,n) = \hat{f}(x)(h,m) = f(h^{-1}x)(m) = 0,$$
since $m \in S_0$. On the other hand, for all $y \in X$,
$$\hat{f}(y)(1,n) = f(y)(m) = 1.$$
Thus $\rho\hat{f}(x) \notin \operatorname{ran} \hat{f}$. ⊣

Claim 2

If ρ^{-1} has type I, then $\rho\hat{f}(x) \notin \operatorname{ran} \hat{f}$ for all $x \in X$.

Proof .:.

Otherwise, there exist $x, y \in X$ such that $\rho\hat{f}(x) = \hat{f}(y)$; and so $\rho^{-1}\hat{f}(y) = \hat{f}(x)$, which contradicts Claim 1. ⊣

So we only need to worry about the elements where it and its inverse have type II or III. Let $\rho_0, \rho_1, \dots \in G$ enumerate the elements of G such that both ρ and ρ^{-1} have type II or III (not necessary both with the same type). Since S_2 is good and every good set can be partitioned into two good sets, we can inductively define *infinite, disjoint* subsets $S_{2,0}, S_{2,1}, \cdots$ of S_2 such that:

- If ρ_i has type II, then every $n \in S_{2,i}$ witnesses this;
- if ρ_i has type III, then every $\rho^{-1}(1,n) \in \mathbb{F}_2 \times \{n\}$ for every $n \in S_{2,i}$.

Constraint 2. For each i, let $h_i : X \to 2^{S_{2,i}}$ be a Borel bijection. Then for every $x \in X$, define $f(x)(n) = h_i(x)(n)$ iff $n \in S_{2,i}$.

So we've defined f on S_0 and S_1, and we're dealing with S_2 and S_3. Next, let $S'_{3,0}, S_{3,0}, S'_{3,1}, S_{3,1}, \dots$, be (finite and possibly empty) disjoint subsets of S_3 such that:

- If ρ_i or ρ_i^{-1} have type II, then $S'_{3,i}$ contains m_{ρ_i} and/or $m_{\rho_i^{-1}}$ provided they're not already included in $S_0 \cup S_1 \cup S_2 \cup \bigcup_{j<i}(S'_{3,j} \cup S_{3,j})$, where m_{ρ_i} is as in Definition 8 A • 3 for type II.
- If ρ_i has type II, then $S_{3,i}$ contains an n which witnesses that ρ_i has type II.
- If ρ_i has type III, then $|S_{3,i}| = 2$ and each $n \in S_{3,i}$ satisfies $\rho_i^{-1}(1,n) \in \mathbb{F}_2 \times \{n\}$.

Again, there is no problem defining $S'_{3,i}$ and $S_{3,i}$.

Constraint 3. We define $f(x)(n) = 0$ if $n \in S_2 \setminus \bigcup_i S_{2,i}$ or $n \in S_3 \setminus \bigcup_i S_{3,i}$.

Finally, we will define $f(x)(n)$ for $n \in S_{3,i}$ by induction on $i \in \omega$. Suppose we have done this for $j < i$. In essence, there is only one candidate.

Claim 3

Suppose that $\rho \in \{\rho_i, \rho_i^{-1}\}$. Then there exists a fixed Borel map $g_\rho : X \to X$ such that for any Borel map $f : X \to Y^{\mathbb{N}}$ satisfying all our previous constraints before step i, if $\rho\hat{f}(x) = \hat{f}(y)$, then $y = g_\rho(x)$.

Proof .:.

First suppose that ρ has type II. Let $\rho = \rho_j$ (possibly $j \geq i$). Then $f(x)(m_\rho)$ has already been defined for each $x \in X$ (being either 0 or else defined at an earlier stage). Thus for every $n \in S_{2,j}$,

$$\rho \hat{f}(x)(1,n) = \hat{f}(x)(\rho^{-1}(1,n)) = \hat{f}(x)(\gamma_n, m_\rho) = f(\gamma_n^{-1}x)(m\rho),$$

for some $\gamma_n \in \mathbb{F}_2$, has already been defined. Suppose that $x, y \in X$ and that $\rho \hat{f}(x) = \hat{f}(y)$. Then for each $n \in S_{2,j}$, by constraint 2,

$$h_j(y)(n) = f(y)(n) = \hat{f}(y)(1,n) = \rho \hat{f}(x)(1,n).$$

Since $h_j : X \to 2^{S_{2,j}}$ is a bijection, there exists at most one such y; namely $g_\rho(x) = h_j^{-1}(n \mapsto \rho \hat{f}(x)(1,n))$, which has already been defined.

Next, suppose that ρ has type III. Then for every $n \in S_{2,j}$,

$$\rho \hat{f}(x)(1,n) = \hat{f}(x)(\rho^{-1}(1,n)) = \hat{f}(x)(\gamma_n, n) = f(\gamma_n^{-1}x)(n),$$

for some $\gamma_n \in \mathbb{F}_2$, has been defined. As above, there exists at most one y such that $\rho \hat{f}(x) = \hat{f}(y)$; namely h_j^{-1} of this function: $y = h_j^{-1}(n \mapsto \rho \hat{f}(x)(1,n))$. ⊣

For each i, let $g_i : X \to X$ be the Borel partial function defined by

$$g_i(x) = y \quad \text{iff} \quad g_{\rho_i}(x) = y \text{ and } g_{\rho_i^{-1}}(y) = x.$$

Therefore,

- If $\rho_i \hat{f}(x) = \hat{f}(y)$, then $g_i(x) = y$.
- g_i is an injection.

To finish the proof, it is enough to complete the construction of f such that for all $x \in X$ and i,

$$\text{either } g_i(x) \; E_\infty \; x \text{ or } \rho_i \hat{f}(x) \neq \hat{f}(g_i(x)). \tag{$\star$}$$

We now continue with step i of the construction. First, suppose that ρ_i has type II. Choose some $n_i \in S_{3,i}$ such that n_i witnesses that ρ_i has type II. Then there exists a $\gamma_i \in \mathbb{F}_2$ such that $\rho_i^{-1}(1, n_i) = \langle \gamma_i, m_{\rho_i} \rangle$. Recall that $\rho_i \hat{f}(x)(1, n_i) = f(\gamma_i^{-1}x)(m_{\rho_i})$ has been defined. Hence to ensure that $\rho_i \hat{f}(x)(1,n_i) \neq \hat{f}(g_i(x))(1,n_i)$, where then $f(g_i(x))(1,n_i) = f(g_i(x)(n)$, it is enough for each $y \in X$ to define

$$f(y)(n_i) = \begin{cases} 1 - f(\gamma_i^{-1} g_i^{-1}(y))(m_{\rho_i}) & \text{if } g_i^{-1}(y) \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$$

And thus we've "killed off" all of type II. More precisely, we have the following.

Claim 4

If ρ_i has type II and $f : X \to Y^{\mathbb{N}}$ satisfies our current constrains, then for all $x \in X$, $\rho_i \hat{f}(x) \notin \operatorname{im} \hat{f}$.

To sum up, for each i, there exists a Borel partial map $g_i : X \to X$ such that

- $\rho_i \hat{f}(x) = \hat{f}(y)$, then $g_i(x) = y$; and
- g_i is an injection.

Moreover, we've dealt with types I and II.

Continuing with step i, suppose ρ_i has type III. Then $|S_{3,i}| = 2$ and for each $n \in S_{3,i}$, there exists a $\gamma_n \in \mathbb{F}_2$ such that $\rho_i^{-1}(1,n) = \langle \gamma_n, n \rangle$. For each $n \in S_{3,i}$ (all two of them), let $g_{i,n} : X \to X$ be the Borel partial function defined by $g_{i,n}(y) = \gamma_n^{-1} g_i^{-1}(y)$. Then if $g_i(x) = y$ and $\rho_i \hat{f}(x) = \hat{f}(y)$, for every $n \in S_{3,i}$,

$$\begin{aligned} f(y)(n) = \hat{f}(y)(1,n) &= \rho_i \hat{f}(x)(1,n) \\ &= \hat{f}(x)(\rho_i^{-1}(1,n)) = \hat{f}(x)(\gamma_n, n) \\ &= f(\gamma_n^{-1}x)(n) = f(g_{i,n}(y))(n) \end{aligned}$$

In other words,

$$f(y)(n) = f(g_{i,n}(y))(n). \tag{†}$$

Until further notice, let $Y = 3$. Let $\mathcal{G}_{i,n}$ be the Borel graph on X such that if $x \neq y$, then x, y are adjacent iff $g_{i,n}(x) = y$ or $g_{i,n}(y) = x$. Then each vertex of $\mathcal{G}_{i,n}$ has degree at most two. Hence there exist Borel 3-colorings $c_{i,n} : X \to 3$. For each $n \in S_{3,i}$ and $y \in X$, define

$$f(y)(n) = c_{i,n}(y) \tag{‡}$$

Claim 5

If ρ_i has type III and $\rho_i \hat{f}(x) = \hat{f}(y)$, then $x \; E_\infty \; y$.

Proof .:.

If $\rho_i \hat{f}(x) = \hat{f}(y)$, then $y = g_i(x)$, the unique candidate, and so $g_i^{-1}(y) = x$ is defined as well as $g_{i,n}^{-1}(y)$ for each $n \in S_{3,i}$. Fix some $n \in S_{3,i}$.

Case 1. Suppose $g_{i,n}(y) = y$. Then $y = \gamma_n^{-1} g_i^{-1}(y) = \gamma_n^{-1} x$ and so they are in the same orbit and thus are E_∞ equivalent.

Case 2. Next suppose $g_{i,n}(y) \neq y$. Then the $c_{i,n}(y) \neq c_{i,n}(g_{i,n}(y))$; and hence (†) and (‡) imply that $\rho_i \hat{f}(x) \neq \hat{f}(y)$. ⊣

This completes the proof of part (ii) of Theorem 8 A • 1. So now we wish to show part (i): that $G \curvearrowright 2^{\mathbb{F}_2 \times \mathbb{N}}$ is measure universal.

8 A • 5. Observation

Suppose that for every Borel probability measure μ on X, there exists a Borel subset $A \subseteq X$ with $\mu(A) = 1$ such that $E_\infty \restriction A$ is Borel reducible to (the orbit equivalence relation of) $G \curvearrowright 2^{\mathbb{F}_2 \times \mathbb{N}}$. Then $G \curvearrowright 2^{\mathbb{F}_2 \times \mathbb{N}}$ is measure universal.

Proof .:.

Let E be a countable Borel equivalence relation on a standard Borel space Z.
Let ν be any Borel probability measure on Z.
Let $\varphi : Z \to X$ be a Borel reduction from E to E_∞.
Let $\mu = \varphi * \nu$, the push-forward.

By the hypothesis, there then exists a Borel subset $A \subseteq X$ with $\mu(A) = 1$ such that $E_\infty \restriction A$ is Borel reducible to $G \curvearrowright 2^{\mathbb{F}_2 \times \mathbb{N}}$. Let $Z_0 = \varphi^{-1}(A)$ by definition of the push-forward. Then $\nu(Z_0) = 1$ and $E \restriction Z_0$ is Borel reducible to $G \curvearrowright 2^{\mathbb{F}_2 \times \mathbb{N}}$. ⊣

Let μ be *any* Borel probability measure on X. Now we use the following lemma to deduce the theorem. Then we prove the lemma.

8 A • 6. Lemma

For any standard Borel space X and Borel partial injections $g_0, g_1 : X \to X$, there exists a Borel subset $A \subseteq X$ with $\mu(A) = 1$ and Borel maps $c_0, c_1 : A \to 2$ such that for all $x \in A$ either

1. there exists $i \in 2$ such that $(g_i \restriction A)(x)$ is undefined;
2. there exists $i \in 2$ such that $g_i(x) = x$; or
3. there exists $i \in 2$ such that $c_i(x) \neq c_i(g_i(x))$.

Assuming Lemma 8 A • 6, we can complete the proof of part (i) as follows. Suppose ρ_i has type III so $|S_{3,i}| = 2$. As before, for each $n \in S_{3,i}$, let $g_{i,n} : X \to X$ be the partial Borel injection defined by

$$g_{i,n}(y) = \gamma_n^{-1} g_{i,n}(y),$$

where $\rho_i^{-1}(1, n) = \langle \gamma_n, n \rangle$. Applying Lemma 8 A • 6, let $A_i \subseteq X$ be Borel with $\mu(A_i) = 1$ and let $c_{i,n} : A_i \to 2$

be as in the lemma. For each $n \in S_{3,i}$ and $y \in X$, we define

$$f(y)(n) = \begin{cases} c_{i,n}(y) & \text{if } \in A_i \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1

If ρ_i has type III and $\rho_i \hat{f}(x) = \hat{f}(y)$ for $x, y \in A_i$, then $x\ E_\infty\ y$.

Proof .:.

The proof is (practically) identical to Claim 5. ⊣

Let $A = \bigcap\{A_i : \rho_i \text{ has type III}\}$. Then $\hat{f} \restriction A$ is a Borel reduction from $E_\infty \restriction A$. This completes the proof of (i) of Theorem 8 A • 1 assuming Lemma 8 A • 6. So finally, we turn to the proof of the lemma.

Proof of Lemma 8 A • 6 .:.

For each $i \in 2$, let $\mathcal{G}_i$ be the Borel graph such that $x \neq y$ are adjacent iff $g_i(x) = y$ or $g_i(y) = x$. Let S_i be the Borel subset of those $x \in X$ such that either

- x has no neighbors in $\mathcal{G}_i$; or
- the connected component of x has an element of degree 1.

Then there exists a Borel 2-coloring of S_i. Hence, to simplify notation, we can suppose that $S_0 = S_1 = \emptyset$, since we can deal with these easily. Let E_i be the countable Borel equivalence relation on X defined by $x\ E_i\ y$ iff x and y lie in the same connected component in $\mathcal{G}_i$. Note that each E_i is aperiodic.

8 A • 7. Lemma

There exists a Borel partition $X = B \sqcup C$ such that $\mu([B]_{E_0}) = \mu([C]_{E_1}) = 1$.

Assuming Lemma 8 A • 7, we can prove Lemma 8 A • 6 as follows. Let $A = [B]_{E_0} \cap [C]_{E_1}$. Then:

- for all $a \in A$, $[a]_{E_0} \cap B \neq \emptyset$;
- for all $a \in A$, $[a]_{E_1} \cap C \neq \emptyset$.

Also $\mu(A) = 1$. Let T_i be the Borel subset of $a \in A$ such that $[a]_{E_i} \not\subseteq A$. For each $a \in T_i$, either

- a has no neighbors in $\mathcal{G}_i \restriction A$; or
- the connected component of a in $\mathcal{G}_i \restriction A$ has an element of degree 1.

Thus there exist Borel 2-colorings $c_i : T_i \to 2$. Hence, it is enough to consider the case when $T_0 = T_1 = \emptyset$; i.e. A is both E_0-invariant and E_1-invariant.

Let $\mathcal{G}_0^*$ be the graph obtained from $\mathcal{G}_0 \restriction A$ by removing edges $\{x, g_0(x)\}$, where $x \in B$; and let $\mathcal{G}_1^*$ be the graph obtained from $\mathcal{G}_1 \restriction A$ by removing edges $\{x, g_1(x)\}$ for $x \in C$. Then every connected component in $\mathcal{G}_i^*$ is either a singleton, or else contains a vertex of degree 1. Hence there exist Borel 2-colorings $c_i : A \to 2$ and of $\mathcal{G}_i^*$. Note that if $x \in A$, then either $x \notin B$ or $x \notin C$. Hence either $\{x, g_0(x)\}$ is an edge of $\mathcal{G}_0^*$ or $\{x, g_1(x)\}$ is an edge of $\mathcal{G}_1^*$. Thus either $c_0(x) \neq c_0(g_0(x))$, or $c_1(x) \neq c_1(g_1(x))$. ⊣

So all that remains is the (second) lemma.

Proof of Lemma 8 A • 7 .:.

Since E_0, E_1 are aperiodic, by The Marker Lemma (4 • 17), there exist decreasing sequences $C_0^i \supseteq C_1^i \supseteq \cdots$ of complete Borel E_i-sections such that $\bigcap_n C_n^i = \emptyset$. Let $C_n = C_n^0 \cup C_n^1$. Then $C_0 \supseteq C_1 \supseteq \cdots$ and $\bigcap_n C_n = \emptyset$. Also, each C_n is a Borel complete section for both E_0 and E_1.

Let A_0 be *any* Borel complete E_0-section (e.g. the whole space). We will define inductively a sequence $\{A_n : n \text{ is even}\}$ of Borel complete E_0-sections and a sequence $\{B_n : n \text{ is odd}\}$ of Borel complete E_1-sections, together with a strictly increasing sequence i_n of natural numbers.

Given A_n, we define B_{n+1} as follows. Since $A_n = A_n \setminus \bigcap_\ell C_\ell = \bigcup_\ell A_n \setminus C_\ell$, it follows that $X = \bigcup_\ell [A_n \setminus C_\ell]_{E_0}$ since A_n was a complete section. Hence there exists an $i_n > i_{n-1}$ such that $\mu([A_n \setminus C_{i_n}]_{E_0}) \geq 1 - (1/2)^n$. We define $B_{n+1} = (A_n \setminus C_{i_n})^{\mathsf{c}}$. Since $B_{n+1} \supseteq C_{i_n}$, B_{n+1} is a Borel complete section for E_1 which satisfies $\mu([B_{n+1}^{\mathsf{c}}]_{E_0}) > 1 - (1/2)^n$. Similarly, given B_n, we define $A_{n+1} = (B_n \setminus C_{i_n})^{\mathsf{c}}$ where $i_n > i_{n-1}$ satsifies $\mu([B_n \setminus C_{i_n}]_{E_1}) > 1 - (1/2)^n$.

Note that A_n^{c} and $B_{n+1}^{\mathsf{c}} = A_n \setminus C_{i_n}$ are disjoint; as are B_n^{c} and A_{n+1}^{c}. Also,

$$A_{n+2}^{\mathsf{c}} = (B_{n+1} \setminus C_{i_{n+1}}) = (A_n \setminus C_{i_n})^{\mathsf{c}} \setminus C_{i_{n+1}} = (A_n^{\mathsf{c}} \cup C_{i_n}) \setminus C_{i_{n+1}} \supseteq A_n^{\mathsf{c}}$$

since $C_{i_{n+1}} \subseteq C_{i_{n-1}} \subseteq A_n$. Similarly, $B_{n+2}^{\mathsf{c}} \supseteq B_n^{\mathsf{c}}$. It follows that $\bigcup_{n \text{ even}} A_n^{\mathsf{c}}$, $\bigcup_{n \text{ odd}} B_n^{\mathsf{c}}$ are disjoint Borel sets such that $\bigcup_{n \text{ odd}} B_n^{\mathsf{c}}$ meets μ-almost every E_0-class and $\bigcup_{n \text{ even}} A_n^{\mathsf{c}}$ meets μ-almost every E_1-class. ⊣

⊣

Section 9. *Odds and Sods*

[class missed, notes transcribed from lecturer's notes]

Our next target is the following consequence of MC.

9 • 1. Theorem

(MC) There exist uncountably many weakly countable universal Borel equivalence relations up to Borel reduction.

We first introduce yet another strong ergodicity notion.

9 • 2. Definition

Suppose that E and F are countable Borel equivalence relations on the standard Borel spaces X and Y, and that μ is an E-invariant, Borel probability measure on X.

E is *F-μ-ergodic* iff for every Borel homomorphism $f : X \to Y$ from E to F, there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z to a single F-class.

Note that if E is F-μ-ergodic and $f : X \to Y$ is a μ-measurable homomorphism from E to F, then there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z to a single F-class.

To see this, let $g : X \to Y$ be a Borel map where $g(x) = f(x)$ for μ-almost every $x \in X$. Then

$$W = \{x \in X : g"[x]_E \text{ is not contained in a single } F\text{-class}\}$$

is an E-invariant Borel subset of X with $\mu(W) = 0$. Hence, after adjusting g on W, we can suppose that g is a Borel homomorphism from E to F. The result follows.

In Section 6, we proved the following theorem.

9 • 3. Theorem

There exists a Borel family $\mathcal{G} = \{G_\alpha : \alpha \in 2^{\mathbb{N}}\}$ of finitely generated groups, each with underlying set $\mathbb{N}$, such that the following conditions hold.

(a) G_α has a normal subgroup $N_\alpha \cong \mathrm{SL}_3(\mathbb{Z})$.

(b) G_α has no nontrivial, finite, normal subgroups.

(c) If $\alpha \neq \beta$, then G_β does not embed into G_α.

For each $\alpha \in 2^{\mathbb{N}}$, consider the shift action $G_\alpha \curvearrowright 2^{G_\alpha} = 2^{\mathbb{N}}$. Then the uniform product probability measure μ on $2^{\mathbb{N}}$ is G_α-invariant and

$$X_\alpha = \{x \in 2^{\mathbb{N}} : g \cdot x \neq x \text{ for all } 1 \neq g \in G_\alpha\}$$

satisfies $\mu(X_\alpha) = 1$. Let E_α be the orbit equivalence relation of $G_\alpha \curvearrowright X_\alpha$. Applying Popa Superrigidity (6 A • 3), we obtain the following theorem.

9 • 4. Theorem

If $\alpha \neq \beta$, then E_β is E_α-μ-ergodic.

In particular, E_α is *not* weakly universal. On the other hand, $\equiv_T \times E_\alpha$ *is* clearly weakly universal.

9 • 5. Theorem

(MC) If $\alpha \neq \beta$, then $(\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha)$.

We will need to work in a forcing extension $V^{\mathbb{P}} \models \mathsf{MA} + \neg\mathsf{CH}$. So we first need to establish some absoluteness results. Recall the definition of MC from Martin's Conjecture (7 • 5). We want to understand the complexity of this statement. For this, we need a parametrization of the Borel relations $R \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

9 • 6. Theorem (Classical Theorem)

There exist subsets $D \subseteq 2^{\mathbb{N}}$ and $P, S \subseteq (2^{\mathbb{N}})^3$ such that:

1. D is Π^1_1, P is Π^1_1, and S is Σ^1_1;
2. If $d \in D$, $P_d = S_d$, where
$$P_d = \{\langle x, y\rangle : \langle d, x, y\rangle \in P\}$$
$$S_d = \{\langle x, y\rangle : \langle d, x, y\rangle \in S\}.$$
For each $d \in D$, let $D_d = P_d = S_d$.
3. $\{D_d : d \in D\} = \{R \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}} : R \text{ is Borel}\}$.

Now consider $F = \{d \in D : D_d \text{ is a function}\}$.

9 • 7. Observation

F is Π^1_1.

Proof .:.

$d \in F$ iff $d \in D$ and

$$\forall x\, \forall y\, \forall z\ \ ([S(d,x,y) \wedge S(d,x,z)] \rightarrow y = z) \dashv$$

As a matter of notation, for each $d \in F$, let F_d be the corresponding Borel map.

It is now easily seen that MC is a Π^1_3 statement. We need to find a less complex formulation.

9 • 8. Definition

(MC′) If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then either:

(a) for all $x \in 2^{\mathbb{N}}$, there exists $x \leq_T y$ such that $f(y) <_T y$; or

(b) for all $x \in 2^{\mathbb{N}}$, there exists $x \leq_T y$ such that $y \leq_T f(y)$.

Note that MC′ is a Π^1_2 statement. Of course, we should want MC to be equivalent to this.

9•9. Theorem

MC $\leftrightarrow$ MC$'$.

Proof .:.

Assume MC and let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel homomorphism from $\equiv_T$ to $\equiv_T$. Suppose there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C to a single class $[r]_{\equiv_T}$. Then for each $x \in 2^{\mathbb{Z}}$, there exists $y \in C$ such that $x \leqslant_T y$ and $f(y) \equiv_T r <_T y$.

Similarly, if there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $z \leqslant_T f(z)$ for all $z \in C$, then for all $x \in 2^{\mathbb{N}}$, there exists $x \leqslant_T y$ such that $y \leqslant_T f(y)$. Hence MC$'$ holds.

Conversely, assume MC$'$ and let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If (a) holds from Definition 9•8, then

$$A = \{y \in 2^{\mathbb{N}} : f(y) <_T y\}$$

is a $\leqslant_T$-cofinal $\equiv_T$-invariant Borel subset of $2^{\mathbb{N}}$. Hence, by Martin's Theorem (7•2), there exists a cone $C \subseteq A$; and by Slaman–Steel (7•6), there exists a cone $D \subseteq C$ such that f maps D into a single $\equiv_T$-class. Similarly, if (b) holds, then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $y \leqslant_T f(y)$ for all $y \in C$. $\dashv$

We also have two more absoluteness results, both immediate consequences of Shoenfield's absoluteness theorem.

9•10. Theorem

If $V \models$ MC and $\mathbb{P}$ is any notion of forcing, then $V^{\mathbb{P}} \models$ MC.

9•11. Theorem

Suppose that E, F are Borel equivalence relations on the standard Borel spaces X, Y and that $f : X \to Y$ is a Borel reduction from E to F. Therefore, if $\mathbb{P}$ is any notion of forcing then $f^{\mathbb{P}} : X^{\mathbb{P}} \to Y^{\mathbb{P}}$ is a Borel reduction from $E^{\mathbb{P}}$ to $F^{\mathbb{P}}$.

[end of class missed]

Let's recall what we've done and what we're working towards.

9•12. Theorem

(MC) There exist uncountably many weakly countable universal Borel equivalence relations up to Borel reduction.

The idea is to have a whole bunch of countable Borel equivalence relations E_α, $\alpha < 2^{\aleph_0}$ such that they are "mutually ergodic", so that are really incompatible. They are not weakly universal, however.

The following are upwards absolute:

- MC
- $E \leqslant_B F$ for Borel equivalence relations E and F

The thing we're trying to prove is the following.

9•13. Theorem

(MC) If $\alpha \neq \beta$, then $(\equiv_T \times E_\beta) \not\leqslant_B (\equiv_T \times E_\alpha)$.

We will use the following consequence of MA + $\neg$CH.

9•14. Theorem

If μ is a Borel probability measure on a standard Borel space X and $Z \subseteq X$ is Σ^1_2, then Z is μ-measurable.

Proof .:.

Since Z is Σ^1_2, there exist Borel subsets A_α, $\alpha < \omega_1$ such that $Z = \bigcup_{\alpha<\omega_1} A_\alpha$. We want to show that Z is μ-measurable.

Let $B \subseteq Z$ be a Borel subset such that $Z \setminus B$ has inner measure 0 (meaning there's no Borel subset of positive measure). Then for each $\alpha < \omega_1$, $\mu(A_\alpha \setminus B) = 0$. By MA + ¬CH, the union $\bigcup_{\alpha<\omega_1} A_\alpha \setminus B = Z \setminus B$ has μ-measure 0. Thus $Z = B \cup (Z \setminus B)$ is μ-measurable. ⊣

Proof of Theorem 9 • 13 .:.

Suppose that for some $\alpha \neq \beta$, $f : 2^{\mathbb{N}} \to X_\beta \to 2^{\mathbb{N}} \times X_\alpha$ is a Borel reduction from $\equiv_{\mathrm{T}} \times E_\beta$ to $\equiv_{\mathrm{T}} \times E_\alpha$. Then we can suppose that MA + ¬CH holds. By upward absoluteness, MC still holds, and f is still a Borel reduction.

Let λ, ρ be the Borel maps such that

$$f(r, x) = \langle \lambda(r, x), \rho(r, x) \rangle.$$

For each $x \in X_\beta$, let $\rho_x : 2^{\mathbb{N}} \to X_\alpha$ be the Borel map defined by $\rho_x(r) = \rho(r, x)$. Then ρ_x is a Borel homomorphism from $\equiv_{\mathrm{T}}$ to E_α. Since E_α isn't weakly universal, by MC, there exists a cone $C_x \subseteq 2^{\mathbb{N}}$ such that ρ_x maps C_x to a single E_α-class; say, $\mathfrak{d}_x$.

Suppose that $y\ E_\beta\ x$ and $r \in C_x$. Then since $\langle r, y \rangle\ (\equiv_{\mathrm{T}} \times E_\beta)\ \langle r, x \rangle$,

$$\rho_y(r) = \rho(r, y)\ E_\alpha\ \rho(x, y) = \rho_x(r),$$

and so $\rho_y(r) \in \mathfrak{d}_x$. Hence if $y\ E_\beta\ x$, then $\mathfrak{d}_y = \mathfrak{d}_x$.

Consider the relation $R \subseteq X_\beta \times X_\alpha$ defined by

$$R(x, z) \quad \text{iff} \quad \exists s\, \forall r\, (s \leqslant_{\mathrm{T}} r \to \rho(r, x)\ E_\alpha\ z).$$

Intuitively, this translates $z \in \mathfrak{d}_x$. Then R is Σ^1_2. By Kondô's theorem, there exists a Σ^1_2-uniformization function $h : X_\beta \to X_\alpha$ for R. Thus $h(x) \in \mathfrak{d}_x$ for all $x \in X_\beta$. If $U \in X_\alpha$ is open,

$$h^{-1}"U = \{x \in X_\beta : \exists y\, (y \in U \wedge h(x) = y)\}$$

and so $h^{-1}"U$ is Σ^1_2. By MA + ¬CH, it follows that $h^{-1}"U$ is μ-measurable. Thus $h : X_\beta \to X_\alpha$ is a μ-measurable homomorphism from E_β to E_α. Since E_β is E_α-μ-ergodic, there exists a Borel $Z \subseteq X_\beta$ with $\mu(Z) = 1$ such that h maps Z to a single E_α-class; say, $\mathfrak{c}$: for every $x \in Z$, $h(x) = \mathfrak{d}_x = \mathfrak{c}$.

For each $x \in Z$, let $\lambda_x : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel map defined by $\lambda_x(r) = \lambda(r, x)$. Then λ_r is a Borel homomorphism from $\equiv_{\mathrm{T}}$ to $\equiv_{\mathrm{T}}$. If $r, s \in C_x$, then $\rho(r, x), \rho(s, x) \in \mathfrak{c}$; and hence $r \equiv_{\mathrm{T}} s$ iff $\lambda_x(r) \equiv_{\mathrm{T}} \lambda_x(s)$. Thus λ_x is a enduces a Borel reduction from $\equiv_{\mathrm{T}} \restriction C_x$ to $\equiv_{\mathrm{T}}$. Hence by MC, there exists a cone $D_x \subseteq [\lambda_x"C_x]_{\equiv_{\mathrm{T}}}$.

In particular, choosing $x, y \in Z$ with $[x]_{E_\beta} \neq [y]_{E_\beta}$, there exist $r \in C_x$ and $s \in C_y$ such that $\lambda_x(r) \equiv_{\mathrm{T}} \lambda_y(s)$. But then $f(r, x)\ (\equiv_{\mathrm{T}} \times E_\alpha)\ f(s, y)$, a contradiction. ⊣

Now we have a concept by Simon, with a name by Kechris.

9 • 15. Definition

A countable group G is *(weakly) action universal* iff there exists a standard Borel G-space such that E^X_G is (weakly) universal.

For notation, if G is countable and X is a standard Borel space,

- $E(G, X)$ is the orbit equivalence relation of $G \curvearrowright X^G$.
- $F(G, x)$ is the free part of $E(G, X)$.

We have the following easy theorems.

9 • 16. Theorem

If the countable group G has a nonabelian free subgroup, then G is action universal.

Proof .:.

Since $\mathbb{F}_2$ embeds in G, it follows that $E_\infty = E(\mathbb{F}_2, 2) \leqslant_{\mathrm{B}} E(G, 2)$. ⊣

9 • 17. Theorem

If G is a countable, amenable group, then G *isn't* action universal.

The proof of Theorem 9 • 17 is delayed. First, we should define what "amenable" means.

9 • 18. Definition

countable group G is *amenable* iff there exists a *finitely additive* probability measure $\mu : \mathcal{P}(G) \to [0,1]$ such that for all $A \subseteq G$ and $g \in G$, $\mu(Ag) = \mu(A)$.

Alternatively, under choice, G is amenable iff for every finite $S \subseteq G$ and $\varepsilon > 0$, there is a nonempty finite $A \subseteq G$ such that for all $s \in S$, $|A \triangle As|/|A| < \varepsilon$. This shows that amenability is absolute.

The two theorems above suggest the possibility of a "dynamical" version of the von Neumann conjecture: is it true that if G is a countable group then the following are equivalent?

i. G is action universal.
ii. G contains a nonabelian free subgroup.

If we replace (i) by "amenable", this is von Neumann's conjecture, and is false. Thomas' conjecture is that the answer is no. Marks' conjecture is that the answer is yes.

We begin working towards a proof of Theorem 9 • 17. First, the following theorem due to Day.

9 • 19. Theorem

If G is a countable group, then the following are equivalent:

1. G is amenable.
2. There exists a sequence of functions $f_n : G \to \mathbb{R}^{\geq 0}$ such that $f_n \in \ell_1(G)$, $\|f_n\|_1 = 1$ and such that for all $g \in G$, $\lim_{n\to\infty} \|f_n - f_n^g\|_1 = 0$ where $f_n^g(h) = f_n(hg)$.

Remark: each f_n can be regarded as a probability measure on G.

Proof of Theorem 9 • 19 .:.

The proof of (ii) from (i) involves functional analysis, and will be skipped. For (i) from (ii), for each $n \in \omega$, define $\mu_n : \mathcal{P}(G) \to [0,1]$ by $\mu_n(A) = \sum_{a\in A} f_n(a)$.

Let U be a nonprincipal ultrafilter on ω. Then $\mu(A) = \lim_U \mu_n(A)$ satisfies our requirements.

Here, for $\langle x_n : n \in \omega\rangle$ a bounded sequence, $\lim_U x_n$ is the unique $\ell \in \mathbb{R}$ such that for each $\varepsilon > 0$, $\{n \in \omega : |r_n - \ell| < \varepsilon\} \in U$. It's not difficult to show such an ℓ exists. $\dashv$

9 • 20. Definition

et E be a countable Borel equivalence relation on X. Then E is 1-*amenable* iff there exist Borel $f_n : E \to \mathbb{R}^{\geq 0}$ such that, letting $f_n^x(y) = f_n(x,y)$,

1. $f_n^x \in \ell_1([x]_E)$ with $\|f_n^x\|_1 = 1$.
2. If $x \mathrel{E} y$, then $\lim_{n\to\infty} \|f_n^x - f_n^y\|_1 = 0$.

9 • 21. Proposition

If G is a countable amenable group and X is a standard Borel G-space, then E_G^X is 1-amenable.

Proof .:.

Let $\langle f_n : n \in \omega\rangle$ witness that G is amenable as in Theorem 9 • 19. For each n, define $g_n : E_G^X \to \mathbb{R}^{\geq 0}$ by

$$g_n(x,z) = \sum_{g\cdot x = z} f_n(g).$$

Clearly, if $x \in X$, then $g_n^x \in \ell_1([x]_{E_G^X})$ and $\|g_n^x\|_1 = 1$. Suppose that $x \mathrel{E_G^X} y$. Then there exists an $h \in G$ such that $hx = y$. For each $z \in [x]_{E_G^X}$, $g_n^x(z) = \sum_{g\cdot x = z} f_n(g)$ and

$$g_n^y(z) = \sum_{gh\cdot x=z} f_n(g) = \sum_{gh\cdot x=z} f_n^{h^{-1}}(gh) = \sum_{g\cdot x=z} f_n^{h^{-1}}(g).$$

Since $\| f_n - f_n^{h^{-1}}\|_1$ tends to 0, it follows that $\|g_n^x - g_n^y\|_1$ also tends to 0. ⊣

In particular, hyperfinite Borel equivalence relations are 1-amenable (because they can be realized by a $\mathbb{Z}$ action, and $\mathbb{Z}$ is amenable). It is not known whether the converse of this statement holds: it's conceivable that hyperfiniteness and 1-amenability are the same.

The converse of Proposition 9 • 21 doesn't hold. And there is an interesting counterexample. Consider the action of $\mathrm{GL}_2(\mathbb{Z})$ (2×2-matrices with determinant ± 1) on $\mathbb{R} \cup \{\infty\}$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} r = \frac{ar+b}{cr+d}.$$

This action is hyperfinite.

9 • 22. Proposition

Suppose G is a countable group and X is a standard Borel space with invariant probability measure μ. If $G \curvearrowright X$ is μ-almost everywhere free, and E_G^X is 1-amenable, then G is amenable.

9 • 23. Corollary

E_∞ isn't 1-amenable.

Proof .:.

$\mathbb{F}_2 \curvearrowright \langle 2^{\mathbb{F}_2}, \mu\rangle$ is μ-almost everywhere free and $\mathbb{F}_2$ *isn't* amenable. ⊣

Proof of Proposition 9 • 22 .:.

Let $\langle \varphi_n : n \in \omega\rangle$ witness that E_G^X is 1-amenable. Define

$$f_n(g) = \int \varphi_n^x(g\cdot x)\, \mathrm{d}\mu(x).$$

Then clearly $f_n \geq 0$. Also

$$\sum_{g\in G} f_n(g) = \sum_{g\in G}\int \varphi_n^x(g\cdot x)\,\mathrm{d}\mu(x) = \int \sum_{g\in G} \varphi_n^x(g\cdot x)\,\mathrm{d}\mu(x) = \int \sum_{y\in[x]} \varphi_n^x(y)\,\mathrm{d}\mu(x),$$

since the action is μ-almost everywhere free. Note that by definition, this is just 1.

Finally, if $h \in G$, then by definition,

$$\begin{aligned}
\|f_n - f_n^h\|_1 &= \sum_{g\in G} |f_n(g) - f_n(gh)| \\
&= \sum_{g\in G} \left| \int \varphi_n^x(g\cdot x)\,\mathrm{d}\mu(x) - \int \varphi_n^x(gh\cdot x)\,\mathrm{d}\mu(x)\right| \\
&= \sum_{g\in G} \left| \int \varphi_n^x(g\cdot x)\,\mathrm{d}\mu(x) - \int \varphi_n^{h^{-1}x}(g\cdot x)\,\mathrm{d}\mu(x)\right| \\
&\leq \sum_{g\in G} \int \left|\varphi_n^x(g\cdot x) - \varphi^{h^{-1}x}(g\cdot x)\right| \,\mathrm{d}\mu(x) \\
&\leq \int \sum_{g\in G} \left|\varphi_n^x(g\cdot x) - \varphi^{h^{-1}x}(g\cdot x)\right| \,\mathrm{d}\mu(x) = \int \left\|\varphi_n^x - \varphi_n^{h^{-1}x}\right\|_1 \,\mathrm{d}\mu(x),
\end{aligned}$$

which goes to 0 as n goes to ∞.

Thus to prove Theorem 9 • 17, it is enough to prove another proposition. Note that it suffices to prove (i), but doing so we need to prove (ii) and (iii).

9 • 24. Proposition

Let E, F be countable Borel equivalence relations on X, Y.

(i) If E is 1-amenable and $F \leqslant_{\mathrm{B}} E$, then F is 1-amenable.

(ii) If E is 1-amenable and $A \subseteq X$ is Borel, then $E \restriction A$ is 1-amenable.

(iii) If $A \subseteq X$ is a complete Borel E-section and $E \restriction A$ is 1-amenable, then E is 1-amenable.

Proof .:.

Assuming (ii) and (iii), we can prove (i) as follows. Let $f : Y \to X$ be a Borel reduction from F to E. Then $A = f"Y$ is a Borel subset of X and there exists a Borel map $g : A \to Y$ such that $f(g(a)) = a$ for all $a \in A$. Also, $B = g"A$ is a Borel complete F-section. Note that $g \restriction A$ is a Borel isomorphism between $E \restriction A$ and $F \restriction B$. Also, by (ii), $E \restriction A$ is 1-amenable and hence so is $F \restriction B$. Since B is a Borel complete F-section, (iii) implies that F is 1-amenable.

(ii) Let $\langle f_n : n \in \omega \rangle$ witness the 1-amenability of E. If A is E-invariant, then $\langle f_n \restriction E \cap A^2 \rangle$ witnesses the 1-amenability of $E \restriction A$. Hence we can assume that A is a Borel complete E-section. Let $\varphi : X \to A$ be a Borel map such that $\varphi(x) \ E \ x$ for all $x \in X$. For $x, y \in A$ with $x \ E \ y$, define $g_n^x(y)$ to be $\Sigma_{z \in \varphi^{-1}(y)} f_n^x(z)$. Then $\langle g_n^x : n \in \omega \rangle$ witnesses the 1-amenability of $E \restriction A$.

(iii) Let $\varphi : X \to A$ be a Borel map such that $\varphi(x) \ E \ x$ for all $x \in X$. For $x, y \in X$ with $x \ E \ y$, define

$$g_n^x(y) = \begin{cases} f_n^{\varphi(x)}(y) & \text{if } y \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle g_n : n \in \omega \rangle$ witnesses the 1-amenability of E. ⊣

9 • 25. Definition

If G is a countable group, then $\mathrm{Sg}(G)$ is the space of subgroups $H \leqslant G$ (a compact subset of 2^G); and $\approx_G$ is the conjugacy relation on $\mathrm{Sg}(G)$:

$$K \approx_G L \quad \text{iff} \quad \exists g \in G \ (gKg^{-1} = L).$$

9 • 26. Theorem

(MC) If G is a countable group, then the following are equivalent.

(i) $\approx_G$ is weakly universal.

(ii) G is weakly action universal.

Proof .:.

That (i) implies (ii) is trivial. So suppose (ii) holds. Let X be a standard Borel G-space such that E_G^X is weakly universal. Suppose $\approx_G$ isn't weakly universal. Sonsider the Borel map $\varphi : X \to \mathrm{Sg}(G)$ given by $\varphi(x) = G_x$ (the stabilizer). Then φ is a Borel homomorphism from E_G^X to $\approx_G$.

Next, let $\psi : 2^{\mathbb{N}} \to X$ be a weak Borel reduction from $\equiv_{\mathrm{T}}$ to E_G^X; and let $\theta = \varphi \circ \psi$. Then θ is a Borel homomorphism from $\equiv_{\mathrm{T}}$ to $\approx_G$. By MC, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that θ maps C to a single $\approx_G$-class. By adjusting ψ if necessary, we can suppose that there exists a fixed $K \leqslant G$ such that $G_{\psi(r)} = K$ for all $r \in C$. For later use, note that $\equiv_{\mathrm{T}} \restriction C$ is weakly universal.

Let $X_0 = \{x \in X : G_x = K\}$. Then we have just seen that $\equiv_{\mathrm{T}} \restriction C \leqslant_{\mathrm{B}}^{\mathrm{w}} E_G^X \restriction X_0$ and so $E_G^X \restriction X_0$ is weakly universal. However, we will next show that $E_G^X \restriction X_0$ is essentially free, a contradiction.

Suppose $x, y \in X_0$ and $x \ E_G^X \ y$. Then there exists a $g \in G$ such that $g \cdot x = y$. Since $gKg^{-1} = gG_xg^{-1} = G_y = K$, it follows that $g \in \mathrm{N}_G(K)$ (the normalizer). If $h \in G$ also satisfies $hx = y$, then $hK = gK$. Hence $E_G^X \restriction X_0$ is the orbit equivalence relation of the associated free Borel action of $\Delta = \mathrm{N}_G(K)/K$. It follows that $E_G^X \restriction X_0$ is essentially free, contradicting that it is weakly universal. ⊣

But what can we prove from ZFC? Without MC, we can just prove the following much weaker version.

9 • 27. Theorem

If $\approx_G$ is essentially free, then G is *not* weakly action universal.

To give an idea of how little is known, consider the following question: is the converse true? Thomas' conjecture (and belief) is "no". Although he changes the conjecture in the last minute of class to be "yes".

We will make use of the following theorem of Hjorth et al.

9 • 28. Theorem

If the countable group G has a nonabelian, free subgroup, then $\approx_G$ is countable universal.

Proof of Theorem 9 • 27 .:.

Suppose G is weakly action universal but that $\approx_G$ is essentially free. Then there exists a countable H and a free standard Borel H-space Z such that $\approx_G \leqslant_{\mathrm{B}} E_H^Z$. Let $\varphi : \mathrm{Sg}(G) \to Z$ be a Borel reduction from $\approx_G$ to E_H^Z. Let L be a finitely generated group with no nontrivial finite normal subgroups such that L doesn't embed into H and let $\Gamma = \mathrm{SL}_3(\mathbb{Z}) \times L$.

Let X be a standard Borel G-space such that E_G^X is weakly universal and let $\psi : 2^\Gamma \to X$ be a weak Borel reduction from $E(\Gamma, 2)$ to E_G^X. Let $\sigma : X \to \mathrm{Sg}(G)$ be the Borel homomorphism defined by $\sigma(x) = G_x$ (the stabilizer). So we have

$$2^\Gamma \xrightarrow{\psi} X \xrightarrow{\sigma} \mathrm{Sg}(G) \xrightarrow{\varphi} Z.$$

Let $\theta : 2^\Gamma \to Z$ be defined by $\theta = \varphi \circ \sigma \circ \psi$. Then θ is a Borel homomorphism from $E(\Gamma, 2)$ to E_H^Z. By Popa Superrigidity (6 A • 3), since L doesn't embed into H, there exists a Borel subset $Y \subseteq 2^\Gamma$ with $\mu(Y) = 1$ such that θ maps Y to a single E_H^Z-class.

Since φ is a Borel reduction, $\sigma \circ \psi$ maps Y to a single $\approx_G$-class. After adjusting ψ if necessary, we can suppose that there exists a single subgroup $K \leqslant G$ such that (the stabilizer) $G_{\psi(y)} = K$ for all $y \in Y$. Let $X_0 = \{x \in X : G_x = K\}$. Then $E_G^X \restriction X_0$ can be realized by the corresponding free action of $\Delta = \mathrm{N}_G(K)/K$. Since $\psi \restriction Y$ is μ-nontrivial, by Popa Superrigidity (6 A • 3), there exists an embedding $\Gamma \hookrightarrow \Delta$. Since $\mathrm{SL}_3(\mathbb{Z}) \leqslant \Gamma$, it follows that Δ contains a nonabelian, free subgroup. It follows that $\mathrm{N}_G(K)$ has a nonabelian, free subgroup. By Theorem 9 • 28, since G has a nonabelian, free subgroup, $\approx_G$ is countable universal and hence *not* essentially free, a contradiction. ⊣

Most questions concerning $\approx_G$ are open. For example, note the following observation to motivate the first question.

9 • 29. Observation

If G, H are countable and there exists a surjective homomorphism $\pi : G \to H$, then $\approx_H \leqslant_{\mathrm{B}} \approx_G$.

Proof .:.

Let $f : \mathrm{Sg}(H) \to \mathrm{Sg}(G)$ be defined by $f(K) = \pi^{-1}"K$. Then f is a Borel reduction from $\approx_H$ to $\approx_G$. ⊣

9 • 30. Open Problem

Suppose $H \leqslant G$. Does it follow that $\approx_H \leqslant_{\mathrm{B}} \approx_G$?

9 • 31. Definition

subgroup $H \leqslant G$ is *malnormal* if $gHg^{-1} \cap H = 1$ for all $g \in G \setminus H$.

For example, let $\mathbb{F}_2 = \langle a, b \rangle < \mathbb{F}_3 = \langle a, b, c \rangle$. Then $\mathbb{F}_2$ is malnormal in $\mathbb{F}_3$.

Note that if H is a malnormal subgroup of G, then clearly $\approx_H \leqslant_{\mathrm{B}} \approx_G$. Also, if $H \leqslant G$ is a counterexample to Open Problem 9 • 30, then G has no nonabelian free subgroups.

9 • 32. Open Problem

Let E be *any* countable Borel equivalence relation. Does there necessarily exist a countable G such that $E \equiv_{\mathrm{B}} \approx_G$?

Really, the question is what kinds of relations can be realized as $\approx_G$ for some G? For now, we at least know there are uncountably many.

9 • 33. Theorem

There exists an uncountable family $\{G_\alpha : \alpha < 2^{\aleph_0}\}$ of finitely generated, nonamenable groups such that if $\alpha \neq \beta$, then $\approx_{G_\alpha}$ and $\approx_{G_\beta}$ are incompatible with respect to $\leqslant_{\mathrm{B}}$. In particular, none are universal. And furthermore, each $\approx_{G_\alpha}$ is essentially free and hence not weakly universal.

To prove this, we require a bit of background.

9 • 34. Definition

If H is any group, then the (restricted) wreath product C_2 wr H is defined as follows. For each $h \in H$, let $C_h = \langle c_h \rangle$ be cyclic of order 2. Then the *base subgroup* is $B = \bigoplus_{h \in H} C_h$, and C_2 wr $H = B \rtimes H$ where $g c_h g^{-1} = c_{gh}$ for $g, h \in H$.

9 • 35. Lemma

If H is a countable group and $G = C_2$ wr H, then $E(H, 2) \leqslant_{\mathrm{B}} \approx_G$.

Proof .:.

For each $A \subseteq H$, let

$$K_A = \bigoplus_{a \in A} C_a \leqslant B \leqslant G.$$

Let $g \in G$ be any element. Then there exist $h \in H$ and $b \in B$ such that $g = hb$. Since B is abelian,

$$g K_A g^{-1} = hbK_A b^{-1} h^{-1} = hK_A h^{-1} = K_{hA}.$$

Thus $A \mapsto K_A$ is a Borel reduction from $E(H, Z)$ to $\approx_G$. ⊣

9 • 36. Corollary

If H is an infinite sum of cyclic groups of order 2 and $G = C_2$ wr H, then $\approx_G$ is nonsmooth and hyperfinite.

Each of our gropus will have the form $G_\alpha = C_2$ wr H_α where H_α is a finitely gnerated, simple, quasi-finite group. In fact, every simple, quasi-finite group is necessarily finitely generated. So the "finitely generated" can be removed, as it's redundant.

To see this, suppose that S is a counterexample. Then S is locally finite. Every infinite, locally finite group has an infinite abelian subgroup (a nontrivial result). Since every proper subgroup is finite, it follows that S must be abelian. But the only abelian, quasi-finite groups are $\mathbb{Z}(p^\infty)$ (which isn't simple) and $\mathbb{Q}$ (which isn't simple).

9 • 37. Lemma

Suppose H is a simple, quasi-finite group and X is a standard Borel H-space. Let $Y = \{x \in X : H_x \neq 1\}$ be the nonfree part of E_H^X. Therefore $E_H^X \restriction Y$ is smooth.

Proof .:.

First let $Z = \{x \in X : H_x = H\}$. Then $E_H^X \restriction Z$ is a clearly smooth. So we can suppose that $Z = \emptyset$. Fix an element $F_{\mathcal{C}}$ of each of the countably many conjugacy classes $\mathcal{C}$ of nontrivial finite subgroups of H. If $x \in Y$, then H_x is a nontrivial finite subgroup of H. Let $\mathcal{C}_x$ be the corresponding cojugacy class containing H_x; and define

$$\pi(x) = \{y \in H \cdot x : H_y = F_{\mathcal{C}_x}\}.$$

We claim that $\pi(x)$ is a nonempty, finite subset of Y. To see that $\pi(x)$ is nonempty, choose $g \in H$ such that $g H_x g^{-1} = F_{\mathcal{C}_x}$ and let $y = g \cdot x$. Then $H_y = g H_x g^{-1} = F_{\mathcal{C}_x}$ and so $y \in \pi(x)$.

Next suppose that $y, z \in \pi(x)$ and let $h \cdot y = z$. Then

$$hF_{\mathcal{C}_x}h^{-1} = hH_yh^{-1} = H_z = F_{\mathcal{C}_x}.$$

Thus $h \in \mathrm{N}_H(F_{\mathcal{C}_x})$. Since H is simple, $\mathrm{N}_H(F_{\mathcal{C}_x})$ is a proper subgroup and hence is finite. Thus $\pi(x)$ is finite. Clearly if $H \cdot x = H \cdot y$, then $\pi(x) = \pi(y)$. Thus $\pi : Y \to Y^{<\omega}$ witnesses that $E_H^X \restriction Y$ is smooth. ⊣

Recall that simple, quasi-finite groups are necessarily finitely generated.

9 • 38. Lemma

Let H be a simple quasi-finite group and let $G = C_2 \text{ wr } H$. Then there exists a free standard Borel space Z such that $E_H^Z \leqslant_{\mathrm{B}} \approx_G$.

Proof .:.

Let $\pi : G \to H$ be the canonical surjection. Then $\mathrm{Sg}(G) = X_0 \sqcup X_1 \sqcup X_1 \sqcup X_2$, where

$$X_0 = \{K \in \mathrm{Sg}(G) : \pi"K = H\}$$
$$X_1 = \{K \in \mathrm{Sg}(G) : \pi"K \text{ is a finite nontrivial subgroup of } H\}$$
$$X_2 = \{K \in \mathrm{Sg}(G) : \pi"K = \{1_H\}\}$$

Let's see how complicated each of these are. Let B be the base group for G as usual: $B = \bigoplus_{h\in H} C_h$.

Claim 1

$\approx_G \restriction X_0$ is smooth.

Proof .:.

Suppose $K \in X_0$ and $g = hb \in G$ be any element where $h \in H$ and $b \in B$. Since $\pi"K = H$, there exists a $c \in B$ such that $k = hc \in K$. It follows that, as B is abelian,

$$g(K \cap B)g^{-1} = hg(K \cap B)g^{-1}h^{-1} = h(K \cap B)h^{-1} = k(K \cap B)k^{-1},$$

as B is normal and k normalizes K, this is just $K \cap B$. Thus $K \cap B \lhd G$. Also since $K/(K \cap B) \cong H$, it follows that K is finitely generated over $K \cap B$ and hence there are only countably many $K' \in X_0$ such that $K' \cap B = K \cap B$.

Let $\equiv$ be the equivalence relation on X_0 defined by $K \equiv K'$ iff $K \cap B = K' \cap B$. Thus $\equiv$ is a smooth (witnessed by sending $K \mapsto K \cap B$) countable Borel equivalence relation. Since $\approx_G \restriction X_0 \subseteq \equiv$, it follows that $\approx_G \restriction X_0$ is smooth. ⊣

Claim 2

$\approx_G \restriction X_1$ is smooth.

Proof .:.

Let $\mathcal{F}$ be a set of representatives of the counjugacy classes of countably many nontrivial finite subgroups of H. For each $F \in \mathcal{F}$, let $X_F = \{K \in \mathrm{Sg}(G) : \pi"K = F\}$. Then $\approx_G \restriction X_1 \leqslant_{\mathrm{B}} \bigsqcup_{F\in\mathcal{F}} \approx_G \restriction X_F$; and so it is enough to show that each $\approx_G \restriction X_F$ is smooth.

Fix some $F \in \mathcal{F}$ and let $K \in X_F$. Since $K/(K \cap B) \cong F$, there are only countably many $K' \in X_F$ such that $K' \cap B = K \cap B$. Hence if $\sim$ is the equivalence relation on X_F defined by

$$K \sim K' \quad \text{iff} \quad \exists h \in \mathrm{N}_H(F)\ (h(K \cap B)h^{-1} = K' \cap B),$$

then $\sim$ is a countable Borel equivalence relation.

Since H is simple, $\mathrm{N}_H(F)$ is a proper subgroup and hence is finite and so $\sim$ is smooth. Hence it is enough to show that $\approx_G \restriction X_F \subseteq \sim$.

Suppose $K, K' \in X_F$ and $gKg^{-1} = K'$. Let $g = hb$, where $h \in H$ and $b \in B$. Then clearly $h \in \mathrm{N}_H(F)$. Also, $K' \cap B = g(K \cap B)g^{-1} = h(K \cap B)h^{-1}$ and so $K \sim K'$. ⊣

By Claim 1 and Claim 2, $\approx_G$ is Borel bireducible with $\approx_G \restriction X_2$. Suppose $K \in X_2$ and $g = hb \in G$, where $h \in H$ and $b \in B$. Then $gKg^{-1} = hKh^{-1}$. Thus $\approx_G \restriction X_2$ can be realized by the corresponding H-action. Let $Z \subseteq X_2$ be the free part of this action. By the previous lemma, $\approx_G \restriction X_2 \sim E_H^Z$ as a simple, quasi-finite group.$\dashv$

We will make use of the following theorem of Ol'shanskii (and folklore).

9 • 39. Theorem

If Γ is a noncyclic, torsion-free, hyperbolic group, then Γ has a family $\{H_\alpha = \Gamma/N_\alpha : \alpha < 2^\omega\}$ of uncountably many non-isomorphic simple, quasi-finite quotients.

To apply Popa Superrigidity (6 A • 3), we want a Kazhdan group. So to use Theorem 9 • 39, we need to be careful about the group we're taking quotients of. Luckily there is a Kazhdan group with these properties. Since the quotients are Kazhdan, we will be able to use Popa Superrigidity (6 A • 3).

9 • 40. Theorem

There exists a noncyclic, torsion-free hyperbolic Kazhdan group.

Proof of Theorem 9 • 33 .:.

Let Γ be a noncyclic, torsion-free, hyperbolic Kazhdan group as per Theorem 9 • 40; and let $\{H_\alpha = \Gamma/N_\alpha : \alpha < 2^\omega\}$ be a family of nonisomorphic, simple, quasi-finite quotients. Then each H_α is also a Kazhdan group.

Let $G_\alpha = C_2 \operatorname{wr} H_\alpha$. Suppose that there exist $\alpha \neq \beta$ such that $\approx_{G_\alpha} \leqslant_B \approx_{G_\beta}$. Since $E(H_\alpha, 2) \leqslant_B \approx_{G_\alpha}$, we have that $E(H_\alpha, 2) \leqslant_B \approx_{G_\beta}$. Let Z be a free standard Borel H_β-space such that $G_\beta \leqslant_B E_{H_\beta}^Z$. Then $E(H_\alpha, 2) \leqslant_B E_H^Z$. So by Popa Superrigidity (6 A • 3), there exists a virtual embedding $\pi : H_\alpha \to H_\beta$. Since H_α is simple, π is an embedding. Since $H_\alpha \not\cong H_\beta$, $\pi(H_\alpha)$ is an infinite, proper subgroup of H_β, which contradicts that H_β is quasi-finite. $\dashv$

Now we discuss the conjecture of Marks. Really we will look at a slight weakening of his actual conjecture (self-described as "ridiculously optimistic").

9 • 41. Definition

Let G be a countable group and let X be a standard Borel G-space.
Then E_G^X is *uniformly universal* iff whenever H is a countable group and Y is a standard Borel H-space, then there exists a Borel reduction $f : Y \to X$ from E_H^Y to E_H^X such that there exists a map (just a function) $u : H \to G$ satisfying

$$f(hy) = u(h)f(y) \tag{*}$$

for all $y \in Y$ and $h \in H$.

In this case, we can suppose that

- $u(1) = 1$;
- if $h \neq h^{-1}$, then $u(h^{-1}) = u(h)^{-1}$ (as we will check).

To see this, suppose $h \neq h^{-1}$. By assumption, for all $y \in Y$,

$$f(hy) = u(h)f(y) \quad \text{ and so } f(y) = u(h)^{-1}f(hy).$$

Let $z \in Y$ be arbitrary and let $y = h^{-1}z$. Then $f(h^{-1}z) = u(h)^{-1}f(z)$.

9 • 42. Open Problem (Marks)

If $E = E_G^X$ is countable universal, then E_G^X is uniformly universal (for each G realizing $E = E_G^X$).

As a convention, from now on, $(2^\omega)^G = \{f : f : G \to 2^\omega\}$.

9 • 43. Example

$E(\mathbb{F}_\omega, 2^\omega)$ is uniformly universal. (Here $\mathbb{F}_\omega$ is the free group on infinitely many generators.)

Proof .:.

Let H be a countable group and Y be a standard Borel H-space. Recall that there exists an injection $\varphi : Y \to (2^\omega)^H$ such that for all $y \in Y$ and $h \in H$, $\varphi(hy) = h\varphi(y)$. Let $\pi : \mathbb{F}_\omega \to H$ be a surjective homomorphism and define $\psi : (2^\omega)^H \to (2^\omega)^{\mathbb{F}_\omega}$ by

$$\psi(x)(g) = x(\pi(g)) \quad x \in (2^\omega)^H, g \in \mathbb{F}_\omega.$$

Then ψ is a Borel reduction from $E(H, 2^\omega)$ to $E(\mathbb{F}_\omega, 2^\omega)$.

Let $u : H \to \mathbb{F}_\omega$ satisfy $\pi(u(h)^{-1}) = h^{-1}$ for $h \in H$. If $x \in (2^\omega)^H$, $h \in H$, and $g \in \mathbb{F}_\omega$, then

$$\psi(hx)(g) = (hx)(\pi(g)) = x(h^{-1}\pi(g)) = x(\pi(u(h)^{-1}g)) = \psi(x)(u(h)^{-1}g) = u(h)\psi(x)(g).$$

Thus $\psi(hx) = u(h)\psi(x)$, as desired. ⊣

We have the following theorem due to Marks.

9 • 44. Theorem

If G is a countable group, then there exists a standard Borel G-space such that E_G^X is uniformly universal iff G has a nonabelian, free subgroup.

Proof .:.

(←) Suppose G has a nonabelian, free subgroup. Then there exists an embedding $\pi : \mathbb{F}_\omega \to G$. Fix some $p_0 \in 2^\omega$, and define $f : (2^\omega)^{\mathbb{F}_\omega} \to (2^\omega)^G$ by

$$f(x)(g) = \begin{cases} x(\pi^{-1}(g)) & \text{if } g \in \operatorname{im}\pi \\ p_0 & \text{otherwise.} \end{cases}$$

Then f is a Borel reduction from $E(\mathbb{F}_\omega, 2^\omega)$ to $E(G, 2^\omega)$; and it is easily checked that $f(hx) = \pi(h)f(x)$ for all $h \in \mathbb{F}_\omega$ and $x \in (2^\omega)^{\mathbb{F}_\omega}$. Taking compositions with the uniformly universal action from Example 9 • 43 yields that $E(G, 2^\omega)$ is uniformly universal.

(→) Suppose there exists a standard Borel G-space X such that E_G^X is uniformly universal. Let $\Gamma = \ast_{i\in\omega}\Gamma_i$ where each $\Gamma_i = \mathbb{F}_2$. Then there exists a Borel reduction $f : (2^\omega)^\Gamma \to X$ from $F(\Gamma, 2^\omega)$ to E_G^X and a map $u : \gamma \to G$ such that for all $y \in (2^\omega)^\Gamma$ and $\gamma \in \Gamma$,

$$f(\gamma y) = u(\gamma)f(y).$$

Applying Theorem 2 • 6 to

$$R = \left\{\langle f(y), y\rangle : y \in (2^\omega)^\Gamma\right\},$$

there exists a partition $(2^\omega)^\Gamma = \bigsqcup_{i\in\omega} A_i$ (seen as the free part) into Borel pieces A_i such that $f \restriction A_i$ is injective. By an extension of Marks' Main Theorem (8 • 17), there exists an $i \in \omega$ and a Γ_i-equivariant, injective, Borel map g going from $(2^\omega)^{\Gamma_i} = (2^\omega)^{\mathbb{F}_2}$ to A_i. Let $\varphi = f \circ g$. Then $\varphi : (2^\omega)^{\mathbb{F}_2} \to X$ is an injection; and if $x \in (2^\omega)^{\mathbb{F}_2}$ and $\gamma \in \mathbb{F}_2$, then

$$\varphi(\gamma x) = f(\gamma g(x)) = u(\gamma)\varphi(x).$$

Let $\mathbb{F}_2 = \langle\alpha, \beta\rangle$. Then, after adjusting u if necessary, we can suppose $u(\alpha^{-1}) = u(\alpha)^{-1}$ and $u(\beta^{-1}) = u(\beta)^{-1}$. Let $a = u(\alpha)$ and $b = u(\beta)$. If $x \in (2^\omega)^{\mathbb{F}_2}$ and $w(\alpha, \beta)$ is a nontrivial reduced word in α, β, then $\varphi(w(\alpha, \beta)x) = w(a, b)\varphi(x)$. Since $w(\alpha, \beta)x \neq x$ and φ is an injection, it follows that $w(a, b) \neq 1$. Then $\langle a, b\rangle$ is a free subgroup of G. ⊣

Finally, we prove the following theorem.

9 • 45. Theorem

There exists a periodic group G of bounded exponent such that $\approx_G$ isn't essentially free.

To prove this, we need some preparation in the form of two theorems of Ol'shanskii.

9 • 46. Theorem

If H is a noncyclic, torsion-free, hyperbolic group, then there exists an integer n_H such that H/H^n is infinite for all odd $n \geq n_H$.

9 • 47. Theorem

For every sufficiently large, odd n, there exists a family $\{G_\alpha : \alpha < 2^\omega\}$ of nonisomorphic, 2-generator, simple groups of exponent n.

So we have our big guns, and can move on to proving Theorem 9 • 45

Proof of Theorem 9 • 45 .:.

Let H be a noncyclic, torsion-free, hyperbolic, Kazhdan group and let n be a sufficiently large, odd integer. Then $K = H/H^n$ is an infinite Kazhdan group of exponent n. Also, let $\{G_\alpha : \alpha < 2^\omega\}$ be a family of nonisomorphic, 2-generator, simple groups of exponent n. Let K be a d-generator group, and let B be the free Burnside group on $d + 2$ generators of exponent n. Then for all $\alpha < 2^\omega$, $K \times G_\alpha$ is a homomorphic image of B and so $E(K \times G_\alpha, 2) \leqslant_{\mathrm{B}} E(B, 2)$.

Claim 1

$E(B, 2)$ isn't essentially free.

Proof .:.

Suppose there exists a countable H and a free standard Borel H-space such that $E(B, 2) \leqslant_{\mathrm{B}} E_H^Y$. For each $\alpha < 2^\omega$, we have that $E(K \times G_\alpha, 2) \leqslant_{\mathrm{B}} E(B, 2) \leqslant_{\mathrm{B}} E_H^Y$. Then Popa Superrigidity (6 A • 3) implies that there exists an embedding $\pi_\alpha : G_\alpha \to H$. Since we have uncountably many such π_α, there exist uncountably many $\alpha \neq \beta$ such that $\pi_\alpha"G_\alpha = \pi_\beta"G_\beta$ and hence $G_\alpha \cong G_\beta$, a contradiction. ⊣

Let $G = C_2 \operatorname{wr} B$. Then $E(G, 2) \leqslant_{\mathrm{B}} \approx_G$ and so $\approx_G$ isn't essentially free. ⊣